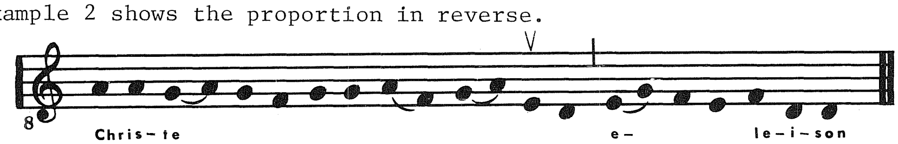
mean precedes the minor mean twice as often as the minor mean precedes the major mean. Example 1 is a section of chant conforming to the M:m proportion.

Example 1*

Example 2 shows the proportion in reverse.



Example 2*

Twenty-one sections have phrase divisions occurring at the arithmetic mean.

The same method was applied to the next larger formal unit, i.e., the three repetitions of each exclamation. In 30 chants there are 90 such units. ϕ is found in 53 (.59) of these units. Where the musical phrase either falls short of the exact mean or extends beyond it, a tolerance of .02 of the total number of pitches was maintained in defining the unit as a golden section.

A performance of an entire chant includes nine sections as shown in Diagram 1. An analysis of the 30 chants revealed that 20 (.66) exhibit the golden section proportion. In more than half of the cases, the mean occurs at the end of the first or at the beginning of the second "Christe eleison."

CONCLUSION

At this stage, these findings tend to establish the presence of the golden section in one of the earliest notated forms of Western music, i.e., the "Kyrie" chants. To establish the presence of the golden section in chants other than the "Kyrie," requires further analysis of the general body of Gregorian chant.

*****

ON FIBONACCI NUMBERS WHICH ARE POWERS

NEVILLE ROBBINS
University of San Francisco, San Francisco, CA 94117

INTRODUCTION

Let $F(n)$, $L(n)$ denote the nth Fibonacci and Lucas numbers, respectively. (This slightly unconventional notation is used to avoid the need for second-order subscripts.) Consider the equation

(0) $F(m) = c^p$,

---

*Source: *Liber Usualis* (Desclee & Co., Tournai [Belb.], 1953), p. 25.

where p is prime and $m > 2$, so that $c > 1$. (The restriction on m eliminates from consideration the trivial solutions which arise because $c^p = c$ if $c = 1$, $c = 0$, or $c = -1$ and p is odd.)

The complete solution of (0) was given for $p = 2$ by J. H. E. Cohn [1] and by O. Wyler [4], and for $p = 3$ by H. London and R. Finkelstein [3]. In this article, we consider (0) for $p \geq 5$. It follows from Theorem 1 that if a nontrivial solution exists, then one exists such that m is odd. In Theorem 2, we give some necessary conditions for the existence of such a solution.

PRELIMINARIES

We will need the following definitions and formulas; r, s denote odd integers such that $(r, s) = 1$.

Definition 1: If q is a prime, then $z(q)$ is the Fibonacci entry point of q, i.e., $z(q) = \min\{m\colon q \mid F(m)\}$.

Definition 2: If q is a prime, then $y(q)$ is the least prime divisor of $z(q)$.

(1) If $(x, y) = 1$ and $xy = z^n$, then $x = u^n$ and $y = v^n$, where $(u, v) = 1$ and $uv = z$.

(2) $F(2n) = F(n)L(n)$.

(3) $\big(F(n), L(n)\big) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$

(4) $F(n) = 2r \leftrightarrow n \equiv 3 \pmod 6 \leftrightarrow L(n) = 4s$.

(5) If $(x, y) = 1 < x$, and $x^m y = z^n$, then $n \mid m$.

(6) $F(n) = 2^k r,\ k > 1 \leftrightarrow k \geq 3,\ 3*2^{k-2} \mid n \leftrightarrow L(n) = 2s$.

(7) $2 \mid F(n) \leftrightarrow 3 \mid n$.

(8) $3 \mid F(n) \leftrightarrow 4 \mid n$.

(9) $\big(F(n), F(kn)/F(n)\big) \mid k$.

(10) t odd $\rightarrow \big(F(t), F(3t)/F(t)\big) = 1$.

(11) $t > 0 \rightarrow F(t) < F(6t)$.

(12) $q \mid F(m) \rightarrow z(q) \mid m$.

(13) $F(2n + 1) = F(n)^2 + F(n + 1)^2$.

(14) c, n odd $\rightarrow c^n \equiv c \pmod 8$.

Remarks: (1) through (8) and (11) through (14) are elementary and/or well-known; for proof of (9), see [2], Lemma 16; (10) follows from (8) and (9).

THE MAIN THEOREMS

For a given prime, p, let $m = m(p) > 2$ be the least integer such that, by assumption, (0) has a nontrivial solution. By inspection,

$$m(2) = 12 \quad \text{and} \quad m(3) = 6.$$

Theorem 1: If $m = 2n > 2$ is the least integer such that $F(m) = c^p$, where p is prime, then either (i) $m = 6$, $p = 3$, or (ii) $m = 12$, $p = 2$.

Proof:

Case 1—If $n \not\equiv 0 \pmod 3$, then by hypothesis, (1), (2), and (3), we have

$F(n) = b^p$. If $b > 1$, we have a contradiction, since $n < m$. If $b = 1$, then hypothesis $\to n = 2 \to m = 4 \to F(m) = 3$, a contradiction.

Case 2—If $n \equiv 3 \pmod 6$, then $(4) \to F(n) = 2r$, $L(n) = 4s$, with rs odd. Now hypothesis and $(2) \to F(m) = 8rs = c^p$, so that $(5) \to p|3 \to p = 3$. By [3], we must have $c = 2$, $n = 3$, $m = 6$.

Case 3—If $n \equiv 0 \pmod 6$, let $n = n_0 = 2^j 3^k t$, where $j, k \geq 1$ and $(6, t) = 1$. Let $n_i = 2^{-i} n_0$ for each i such that $1 \leq i \leq j$. Let $h_0 = n_j = 3^k t$, and let $h_i = 3^{-i} h_0$ for each i such that $1 \leq i \leq k$, so that $t = h_k$. By (6), we have $F(n) = 2^{2+j} r$, $L(n) = 2s$, where rs is odd and $(r, s) = 1$. Now hypothesis, (1), and (2) imply $r = r_0^p$, $s = s_0^p$, with $r_0 s_0$ odd and $(r_0, s_0) = 1$. Therefore, $F(n) = F(n_0) = 2^{2+j} r_0^p$, $L(n) = L(n_0) = 2s_0^p$, $r_0 s_0 = c$. Since $n_i = 2n_{i+1}$, we may repeat our reasoning to obtain $F(n_i) = F(n_{i+1})L(n_{i+1}) = 2^{2+j-i} r_i^p$, $L(n_i) = 2s_i^p$ for $i = 0, 1, 2, \ldots, j - 1$. By (4) we have $F(h_0) = F(n_j) = 2r_j^p$, $L(n_j) = 4s_j^p$; moreover, $r_i s_i = r_{i-1}$ is odd and $(r_i, s_i) = 1$ for $i = 1, 2, 3, \ldots, j$. Now, let $r_j = u_0$, so that $F(h_0) = 2u_0^p$. We have $F(h_{i-1}) = F(h_i) * F(h_{i-1})/F(h_i)$ for $i = 1, 2, 3, \ldots, k$. By (7), (10), and (1), if $i < k$, we have $F(h_i) = 2u_i^p$, $F(h_{i-1})/F(h_i) = v_i^p$; if $i = k$, we have $F(t) = F(h_k) = u_k^p$, $F(h_{k-1})/F(h_k) = 2v_k^p$; moreover, $(u_i, v_i) = 1$ and $u_i v_i = u_{i-1}$ is odd for $i = 1, 2, 3, \ldots, k$.

But $(11) \to F(t) < F(6t) \leq F(n) < F(m) = c^p \to u_k = 1 \to t = 1$. If $k \geq 2$, then $F(h_{k-2})/F(h_{k-1}) = F(9)/F(3) = 17 = v_{k-1}^p \to p = 1$, a contradiction. Hence, $k = 1$, $h_0 = n_j = 3$. If $j \geq 2$, then $L(n_{j-2}) = L(12) = 322 = 2s_{j-2}^p \to s_{j-2}^p = 161 \to p = 1$, a contradiction. Therefore, $k = j = 1$, $n = 6$, $m = 12$, $p = 2$.

Corollary: If (0) has a nontrivial solution for $p \geq 5$, then it has a nontrivial solution such that m is odd.

Proof: The proof follows directly from Theorem 1.

Theorem 2: If $F(m) = c^p > 1$, where the prime $p \geq 5$, and m is odd, then either (i) $m \equiv \pm 1 \pmod{12}$ and $c \equiv 1 \pmod 8$ or (ii) $m \equiv \pm 5 \pmod{12}$ and $c \equiv 5$ (mod 8; furthermore, if q is any prime factor of c, then $y(q) \geq 5$, so that

$$q \in \{5, 13, 37, 73, 89, 97, 113, 149, 157, \ldots\}.$$

Proof: If $2|c$, then $2^p | c^p \to 2^p | F(m)$, so that by (6), $3 * 2^{p-2} | m$, contradicting hypothesis. Now c is odd, so that $F(m)$ is odd, and by (7), $3 \nmid m$. Therefore, $m \equiv \pm 1$ or $\pm 5 \pmod{12}$. If q is any prime factor of c, then

$$(12) \to y(q) | z(q) | m.$$

Since $(6, m) = 1$, we must have $y(q) \geq 5$.

Case 1—If $m = 12t \pm 1$, then $(13) \to F(m) = F(6t)^2 + F(6t \pm 1)^2 = c^p$. Now, $F(6t) \equiv 0 \pmod 8$ and $F(6t \pm 1)$ is odd, so $F(6t \pm 1)^2 \equiv 1 \pmod 8$. Therefore, $c^p \equiv 0 + 1 \equiv 1 \pmod 8$, and (14) implies $c \equiv 1 \pmod 8$.

Case 2—If $m = 12t \pm 5$, then $(13) \to F(m) = F(6t \pm 3)^2 + F(6t \pm 2)^2 = c^p$. Now, $F(6t \pm 3) \equiv 2 \pmod 8$ and $F(6t \pm 2)$ is odd, so $F(6t \pm 2)^2 \equiv 1 \pmod 8$. Therefore, $c^p \equiv 4 + 1 \equiv 5 \pmod 8$, and (14) implies $c \equiv 5 \pmod 8$.

REFERENCES

1. J. H. E. Cohn, "Square Fibonacci Numbers, Etc.," *The Fibonacci Quarterly*, Vol. 2, No. 2 (1964), pp. 109-113.
2. J. H. Halton, "On the Divisibility Properties of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 4, No. 3 (1966), pp. 217-240.
3. H. London & R. Finkelstein, "On Fibonacci and Lucas Numbers Which Are Perfect Powers," *The Fibonacci Quarterly*, Vol. 7 (1969), pp. 476-481.
4. O. Wyler, *American Math. Monthly*, Vol. 71 (1964), pp. 221-222.

*****