THE NORMAL MODES OF A HANGING OSCILLATOR OF ORDER N

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ABSTRACT

The normal frequencies are computed for a system of N identical oscillators, each hanging from the one above it, and the highest oscillator hanging from a fixed point. These frequencies are obtainable from the roots of the Chebyshev polynomials of the second kind.

A massless spring of harmonic constant k is suspended from a fixed point, and from it is suspended a mass m. This system will oscillate with an angular frequency $\omega_0 = (k/m)^{1/2}$. If N such oscillators are thus suspended, each one from the one above it, we will call this system a hanging oscillator of order N.

The Lagrangian for this system is

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = \frac{1}{2}m\sum_{i=1}^{N}\dot{q}_1^2 - \frac{1}{2}kq_1^2 - \frac{1}{2}k\sum_{i=2}^{N}(q_i - q_{i-1})^2, \tag{1}$$

where q_i is the displacement of the ith mass from its equilibrium position. This Lagrangian can also be written in the language of matrix algebra as

$$L = \frac{1}{2}m\dot{q}^T T\dot{q} - \frac{1}{2}m\omega_0^2 q^T Uq \tag{2}$$

where q and $\dot{q}$ are, respectively, the column vectors $\mathrm{col}(q_1, q_2, \ldots, q_N)$ and $\mathrm{col}(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N)$. It is obvious that $T = I$, where I is the $N \times N$ identity matrix. For U, we state the following theorem.

Theorem 1: $u_{ii} = 2$ and $u_{i,i+1} = u_{i+1,i} = -1$ for $i = 1, 2, \ldots, N-1$; $u_{NN} = 1$, and all other values of u_{ij} are zero.

This can be demonstrated by mathematical induction. It is obvious for $N = 1$. For $N = n$ the last two terms in (1) are

$$-\frac{1}{2}m\omega_0^2(q_{n-1} - q_{n-2})^2 - \frac{1}{2}m\omega_0^2(q_n - q_{n-1})^2. \tag{3}$$

From these terms come the matrix elements $u_{n-1,n-1} = 2$, $u_{n-1,n} = u_{n,n-1} = -1$, $u_{nn} = 1$. For $N = n + 1$, these terms are added to (1):

$$\frac{1}{2}m\dot{q}_{n+1}^2 - \frac{1}{2}m\omega_0^2(q_{n+1} - q_n)^2. \tag{4}$$

The matrix element u_{nn} is now increased to 2, and the additional elements $u_{n,n+1} = u_{n+1,n} = -1$, $u_{n+1,n+1} = 1$ now appear in the new $(n+1) \times (n+1)$ matrix U.

The characteristic function for this problem is $\det(-m\omega^2 T + m\omega_0^2 U)$. If we let $x = \omega/\omega_0$, then the normal frequencies for a hanging oscillator of order N are given by the N positive roots of the polynomial $\det(-x^2 I + U) = 0$. Each of the diagonal elements of this determinant is $(-x^2 + 2)$ except for the last, which is $(-x^2 + 1)$. The only other nonzero elements are those immediately next to the diagonal elements; they are each -1.

In the solution of this problem, the Fibonacci polynomials [1] will be useful. These polynomials are defined by the recurrence relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \text{ where } F_1(x) = 1 \text{ and } F_2(x) = x.$$

By repeated application of this recurrence relation, we can prove:

Theorem 2: $F_{n+4}(x) = (x^2 + 2)F_{n+2}(x) - F_n(x)$.

Theorem 2 can be used to prove:

Theorem 3: The characteristic function for the hanging oscillator of order N is

$$(m\omega_0^2)^N F_{2N+1}(ix). \tag{5}$$

The factor $(m\omega_0^2)^N$ comes out of the determinant, leaving $\det(-x^2 I + U)$. Theorem 3 thus reduces to the evaluation of the determinant

$$|V| = \begin{vmatrix} -x^2+2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -x^2+2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & -1 & 0 \\ 0 & 0 & \cdots & -1 & -x^2+2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & -x^2+1 \end{vmatrix} \tag{6}$$

to show that it equals $F_{2N+1}(ix)$.

If $N = 1$, Theorem 3 obviously holds, and $F_3(x) = -x^2 + 1$. Let us assume that the determinant (6) is $F_{2n+1}(ix)$ for $N = n$. Then for $N = n + 1$ we will expand the determinant by minors. It is v_{11} times the minor of v_{11} minus v_{12} times the minor of v_{12}. But the minor of $v_{11} = -x^2 + 2$ is the characteristic function $F_{2n+1}(ix)$ for $N = n$. The minor of v_{12} is (-1) times the characteristic function $F_{2n-1}(ix)$ for $N = n - 1$. The determinant (6) is therefore

$$(-x^2 + 2)F_{2n+1}(ix) - F_{2n-1}(ix),$$

which by Theorem 2 is equal to

$$F_{2(n+1)+1}(ix).$$

Theorem 3 is thus proved by mathematical induction.

Theorem 4: The characteristic frequencies of a hanging oscillator of order N are

$$\omega_0 x_j = \omega_j = 2\omega_0 \cos\frac{j\pi}{2N+1}, \quad j = 1, 2, \ldots, N. \tag{7}$$

The Fibonacci polynomials and the Chebyshev polynomials of the second kind $U_N(x)$ are related by [2]:

$$F_{N+1}(x) = i^{-N}U_N\left(\frac{1}{2}ix\right). \tag{8}$$

The Fibonacci polynomials of imaginary argument then become:

$$F_{N+1}(ix) = i^{-N}U_N\left(-\frac{1}{2}x\right) \tag{9}$$

and the Fibonacci polynomials of interest in this problem become:

$$F_{2N+1}(ix) = (-1)^N U_{2N}\left(\frac{1}{2}x\right). \tag{10}$$

The roots of the eigenvalue equation obtained by setting the characteristic function (5) equal to zero are those given by (7) [3]. Theorem 4 is thus proved.

Two interesting special cases present themselves when $2N + 1$ is an integral multiple of 3 or of 5.

If $2N + 1 = 3P$, where P is an integer, then the root corresponding to $j = P$ is $\omega = \omega_0$. Thus, one of the normal frequencies is equal to the frequency of a single oscillator in the combination.

If $2N + 1 = 5Q$, where Q is an integer, then the roots corresponding to $j = Q$ and to $j = 2Q$ are, respectively, $\omega = \phi\omega_0$ and $\omega = \phi^{-1}\omega_0$, where

$$\phi = 1.6180339885\ldots$$

is the larger root of $x^2 - x - 1 = 0$, the famous "golden ratio." This ratio occurs frequently in number theory and in the biological sciences [4], but its appearances in physics are very few, and usually seem contrived [5].

The coordinates q as functions of time are given by [6]

$$q_j(t) = \sum_{k=1}^{N} a_j k \cos(\omega_k t - \delta_k) \tag{11}$$

where a_{jk} is the kth component of the eigenvector a_j which correspond to the normal frequency ω_j given by (7). These eigenvectors are obtained from the equation

$$m(-\omega_j^2 T + \omega_0^2 U)a_j = m\omega_0^2(-x_j^2 I + U)a_j = 0, \tag{12}$$

and their components therefore obey the following equations:

$$\begin{aligned} &-2a_{j1} \cos\frac{2j\pi}{2N+1} - a_{j2} = 0; \\ &-a_{j,k-2} - 2a_{j,k-1}\cos\frac{2j\pi}{2N+1} - a_{jk} = 0,\ k = 3, 4, \ldots, N. \end{aligned} \tag{13}$$

The components of a_j are therefore

$$\begin{aligned} a_{j2} &= -2a_{j1}\cos\frac{2j\pi}{2N+1}; \\ a_{jk} &= -2a_{j,k-1}\cos\frac{2j\pi}{2N+1} - a_{j,k-2}, \text{ for } k = 3, 4, \ldots, N. \end{aligned} \tag{14}$$

The components a_{jk} can be evaluated from this recursion relation for the Chebyshev polynomials of the second kind [3, p. 782]:

$$U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x) \tag{15}$$

and we obtain

$$a_{jk} = (-1)^{k-1} a_{j1} U_k\left(\cos\frac{2j\pi}{2N+1}\right), \tag{16}$$

where a_{j1} is arbitrary.

If the initial position and velocity of the jth mass are, respectively, X_j and V_j, then the normal coordinates are [6, p. 431]

(17)
$$\begin{aligned}\zeta_k(t) &= Re\sum_{j=1}^{N} ma_{jk}\, e^{i\omega_k t}\left(X_j - \frac{i}{\omega_k}V_j\right)\\ &= Re\sum_{j=1}^{N} m(-1)^{k-1}a_{j1}U_k\left(\cos\frac{2k\pi}{2N+1}\right)\exp\left[2i\omega_0 t\cos\frac{k\pi}{2N+1}\right]\\ &\qquad\times\left(X_j - \frac{iV_j}{2\omega_0\cos\frac{k\pi}{2N+1}}\right)\end{aligned}$$

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CONGRUENCES FOR CERTAIN FIBONACCI NUMBERS

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The purpose of this note is to prove some of the well-known congruences for the Fibonacci numbers U_p and U_{p-1}, where p is prime and $p \equiv \pm 1 \pmod 5$. We also prove a congruence which is analogous to

$$U_n = \frac{\alpha^\mu - \beta^\mu}{\alpha - \beta}, \text{ where } \alpha \text{ and } \beta \text{ are the roots of } x^2 - x - 1 = 0.$$

We start by considering the congruence

(1) $x^2 - x - 1 \equiv 0 \pmod p$, which can also be written

(2) $y^2 \equiv 5 \pmod p$,

on putting $2x - 1 = y$.

It is well known that 5 is a quadratic residue of primes of the form $5m \pm 1$ and a quadratic nonresidue of primes of the form $5m \pm 3$. Therefore, (2) has a solution p if p is a prime and $p \equiv \pm 1 \pmod 5$.

It also has $-y$ as a solution, and these solutions are different in the sense that

$$y \not\equiv -y \pmod p.$$

This obviously gives two different solutions x_1 and x_2 of (1).