ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR. S.E., ALBUQUERQUE, NEW MEXICO 87108. Each solution (or problem) should be submitted on a separate sheet of paper. Preference will be given to solutions (or problems) typed, double-spaced, in the format used below. Solutions (or problems) should be received no later than four months following (or prior to) the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-406 *Proposed by Wray G. Brady, Slippery Rock State College, PA.*

Let $x_n = 4L_{3n} - L_n^3$ and find the greatest common divisor of the terms of the sequence $x_1, x_2, x_3, \ldots$.

B-407 *Proposed by Robert M. Giuli, Univ. of California, Santa Cruz, CA.*

Given that

$$\frac{1}{1 - x - xy} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} x^n y^k$$

is a double ordinary generating function for a_{nk}, determine a_{nk}.

B-408 *Proposed by Lawrence Somer, Washington, D.C.*

Let $d \in \{2, 3, \ldots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. For $d = 2$ and $z(p)$ an even integer $2k$, it was shown in B-386 that

$$F_{n+1}G_{n+k} \equiv F_n G_{n+k+1} \pmod{p}.$$

Establish a generalization for $d \geq 2$.

B-409 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Let $P_n = F_n F_{n+a}$.

Must $P_{n+6r} - P_n$ be an integral multiple of $P_{n+4r} - P_{n+2r}$ for all nonnegative integers a and r?

B-410 *Proposed by M. Wachtel, Zürich, Switzerland.*

Some of the solutions of

$$5(x^2 + x) + 2 = y^2 + y$$

in positive integers x and y are:

$$(x,y) = (0,1),\ (1,3),\ (10,23),\ (27,61).$$

Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n,y_n). Also find $\lim(x_{n+1}/x_n)$ and $\lim(x_{n+2}/x_n)$ as $n \to \infty$ in term of

$$a = (1 + \sqrt{5})/2.$$

B-411 *Proposed by Bart Rice, Crofton, MD.*

Tridiagonal n by n matrices $A_n = (a_{ij})$ of the form

$$a_{ij} = \begin{cases} 2a \ (a \text{ real}) \text{ for } j = i \\ 1 \quad \text{for } j = i \pm 1 \\ 0 \quad \text{otherwise} \end{cases}$$

occur in numerical analysis. Let $d_n = \det A_n$.

(i) Show that $\{d_n\}$ satisfies a second-order homogeneous linear recursion.
(ii) Find closed-form and asymptotic expressions for d_n.
(iii) Derive the combinatorial identity

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1}(-x)^k = (x+1)^{(n-1)/2}\frac{\sin rn}{\sin r},$$

$$x > 0,\ r = \tan^{-1}\sqrt{x}.$$

SOLUTIONS

Lucky L Units Digit

B-382 *Proposed by A. G. Shannon, N.S.W. Inst. of Technology, Australia.*

Prove that L_n has the same last digit (i.e., units digit) for all n in the infinite geometric progression 4, 8, 16, 32, ...

Note: Several solvers pointed out that the subscript n was missing from the L_n.

Solution by Lawrence Somer, Washington, D.C.

I present two solutions, the first of which is more direct.

First Solution: Note that

$$L_n^2 = (a^n + b^n)^2 = a^{2n} + b^{2n} + 2(ab)^n = L_{2n} + 2(-1)^n.$$

We now proceed by induction. $L_4 = 7$. Now assume

$$L_{2^n} \equiv 7 \pmod{10},\ n \geq 2.$$

Then

$$L_{2^{n+1}} + 2(-1)^{2^n} = L_{2^{n+1}} + 2 = L_{2^n}^2 \equiv 7^2 \equiv 9 \pmod{10}.$$

Thus,

$$L_{2^{n+1}} \equiv 9 - 2 \equiv 7 \pmod{10}$$

and we are done.

Second Solution: Note that the Lucas sequence has a period modulo 10 of 12. Now $\{2^n\}_{n=2}^{\infty} \pmod{12}$ is of the form 4, 8, 4, 8, ... But L_4 and L_8 both end in 7. Thus, we are done.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, Bob Prielipp, Sahib Singh, Charles W. Trigg, Gregory Wulczyn, and the proposer.

Reappearance

B-383 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.*

Solve the difference equation

$$U_{n+2} - 5U_{n+1} + 6U_n = F_n.$$

Note: Bob Prielipp and Sahib Singh point out that B-383 is a rerun of B-370.

Solvers in addition to those of B-370 are Ralph Garfield, Lawrence Somer, and Gregory Wulczyn.

A Recursion for F_{2n}^4 or F_{2n+1}^4

B-384 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA.*

Establish the identity

$$F_{n+10}^4 = 55(F_{n+8}^4 - F_{n+2}^4) - 385(F_{n+6}^4 - F_{n+4}^4) + F_n^4.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA.

It suffices to prove that

(1) $$F_{n+10}^4 - F_n^4 = 55(F_{n+8}^4 - F_{n+2}^4) - 385(F_{n+6}^4 - F_{n+4}^4).$$

Factoring the difference of squares, one sees that (1) follows from the two formulas:

(2) $$(F_{n+10}^2 - F_n^2)/55 = (F_{n+8}^2 - F_{n+2}^2)/8 = F_{n+6}^2 - F_{n+4}^2;$$

(3) $$F_{n+10}^2 + F_n^2 = 8(F_{n+8}^2 + F_{n+2}^2) - 7(F_{n+6}^2 + F_{n+4}^2).$$

Each of (2) and (3) can be established using the Binet formulas, $a^2 = a + 1$, $a^4 = 3a + 2$, etc., and the corresponding formulas for powers of b.

Also solved by Paul S. Bruckman and the proposer.

Counting Some Triangular Numbers

B-385 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let $T_n = n(n+1)/2$. For how many positive integers n does one have both $10^6 < T_n < 2 \cdot 10^6$ and $T_n \equiv 8 \pmod{10}$?

Solution by Lawrence Somer, Washington, D.C.

By inspection, $T_n \equiv 8 \pmod{10}$ if and only if

(1) $$n \equiv 7 \pmod{20} \text{ or } n \equiv 12 \pmod{20}.$$

Now, $10^6 < T < 2 \cdot 10^6$ if and only if

(2) $$-1/2 + \sqrt{2{,}000{,}000.25} < n < -1/2 + \sqrt{4{,}000{,}000.25}$$

or $1414 \leq n \leq 1999$. There are 58 integers satisfying conditions (1) and (2). The answer is thus 58.

Also solved by Paul S. Bruckman, Sahib Singh, Charles W. Trigg, Gregory Wulczyn, and the proposer.

Elusive Generalization

B-386 *Proposed by Lawrence Somer, Washington, D.C.*

Let p be a prime and let the least positive integer m with $F_m \equiv 0 \pmod{p}$ be an even integer $2k$. Prove that

$$F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.$$

Generalize to other sequences, if possible.

Solution by Paul S. Bruckman, Concord, CA.

The following formula may be readily verified from the Binet definitions:

(1) $$F_{n+1}L_{n+k} - F_nL_{n+k+1} = (-1)^nL_k.$$

Since $F_{2k} = F_kL_k \equiv 0 \pmod{p}$ and $2k$ is the least positive integer m such that $F_m \equiv 0 \pmod{p}$, thus $F_k \not\equiv 0 \pmod{p}$, which implies $L_k \equiv 0 \pmod{p}$. From (1), we see that this, in turn, implies

(2) $$F_{n+1}L_{n+k} \equiv F_nL_{n+k+1} \pmod{p}.$$

The desired generalization to other sequences appears to be elusive.

Editor's note: For one generalization, see B-408, proposed in this issue.

Also solved by Sahib Singh, Gregory Wulczyn, and the proposer.

One's Own Infinitude

B-387 *Proposed by George Berzsenyi, Lamar Univ., Beaumont, TX.*

Prove that there are infinitely many ordered triples of positive integers (x,y,z) such that

$$3x^2 - y^2 - z^2 = 1.$$

Editor's note: An infinite number of solutions were produced with $y = z + 2$ by Paul. S. Bruckman, with $y = z$ and with $z = 1$ by Bob Prielipp, with $x = z$ by Sahib Singh, with $z \in \{1, 5, 11, 25\}$ by Gregory Wulczyn, and with $(x^2, y^2, z^2) = (F_{2n+2}w + 1, F_{2n}w + 1, F_{2n+4}w + 1)$, where $w = 4F_{2n+1}F_{2n+2}F_{2n+3}$, by the proposer. Of these, the following was chosen for publication because of its bibliographic and historical references.

Solution by Bob Prielipp, Univ. Of Wisconsin-Oshkosh, WI.

We will show that there are infinitely many ordered triples of positive integers $(x,y,1)$ such that

$$3x^2 - y^2 - 1^2 = 1.$$

The preceding equation is equivalent to $y^2 - 3x^2 = -2$. To assist us in finding all of its solutions, we will employ the following results:

(1) If D and N are positive integers, and if D is not a perfect square, the equation $u^2 - Dv^2 = -N$ has a finite number of classes of solutions. If (u^*, v^*) is the fundamental solution of the class K, we obtain all the solutions (u, v) of K by the formula

$$u + v\sqrt{D} = (u^* + v^*\sqrt{D})(s + t\sqrt{D}),$$

where (s, t) runs through all solutions of $s^2 - Dt^2 = 1$, including $(\pm 1, 0)$.

(2) If p is a prime, and if the equation $u^2 - Dv^2 = -p$ is solvable, it has one or two classes of solutions, according as the prime p divides $2D$ or not. [See Nagell, *Introduction to Number Theory* (2nd ed.; New York: Chelsea Publishing Company, 1964), pp. 204-208.]

By (2), there is only one class of solutions of the equation

$$y^2 - 3x^2 = -2.$$

The fundamental solution is (1,1). The fundamental solution of the equation $y^2 - 3x^2 = 1$ is (2,1). So, all positive integer solutions of $y^2 - 3x^2 = -2$ are given by the formula

$$y + x\sqrt{3} = (1 + \sqrt{3})(2 + \sqrt{3})^n, \; n = 0, 1, 2, 3, \ldots .$$

Thus, the first six positive integer solutions (y, x) of $y^2 - 3x^2 = -2$ are

(1,1), (5,3), (19,11), (71,41), (265,153), and (989,571).

The corresponding six positive integer solutions (x, y, z), with $z = 1$, of the equation $3x^2 - y^2 - z^2 = 1$ are

(1,1,1), (3,5,1), (11,19,1), (41,71,1), (153,165,1), and (571,989,1).

It may be of interest to note that, in his famous *Measurement of a Circle*, Archimedes determines that $3\frac{1}{7} > \pi > 3\frac{10}{71}$ and in deducing these inequalities he uses $\frac{1351}{780} > \sqrt{3} > \frac{265}{153}$. It can be shown that these good approximations to $\sqrt{3}$ satisfy the equations $a^2 - 3b^2 = 1$ and $a^2 - 3b^2 = -2$, respectively, so that Archimedes knew at least some solutions of these equations.

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