SPECIAL RECURRENCE RELATIONS ASSOCIATED WITH THE SEQUENCE $\{w_n(a, b; p, q)\}$*

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1. INTRODUCTION

There are three parts to this paper, the link being $\{w_n\}$, defined below in (1.1). In the first, a lacunary recurrence relation is developed for $\{w_n\}$ in (2.3) from a multisection of a related series. Then a functional recurrence relation for $\{w_n\}$ is investigated in (3.2). Finally, a q-series recurrence relation for $\{w_n\}$ is included in (4.5).

The generalized sequence of numbers $\{w_n\}$ is defined by

(1.1) $$w_n = pw_{n-2} - qw_{n-2}\ (n \geq 2),\ w_0 = a,\ w_1 = b,$$

where p,q are arbitrary integers. Various properties of $\{w_n\}$ have been developed by Horadam in a series of papers [4, 5, 6, 7, and 8].

We shall have occasion to use the "fundamental numbers," $U_n(p,q)$, and the "primordial numbers," $V_n(p,q)$, of Lucas [10]. These are defined by

(1.2) $$U_n(p,q) \equiv w_n(0,1;p,q),$$

(1.3) $$V_n(p,q) \equiv w_n(2,p;p,q).$$

For notational convenience, we shall use

(1.4) $$U_n(p,q) \equiv U_n \equiv u_{n-1} = (\alpha^n - \beta^n)/(\alpha - \beta),$$

(1.5) $$V_n(p,q) \equiv V_n \equiv v_{n-1} = \alpha^n + \beta^n,$$

where α,β are the roots of $x^2 - px + q = 0$.

2. LACUNARY RECURRENCE RELATION

We define the series $w(x)$ by

(2.1) $$w(x) = w_1(x) = \sum_{n=0}^{\infty} w_n x^n,$$

the properties of which have been examined by Horadam [4].

If r is a primitive mth root of unity, then the kth m-section of $w(x)$ can be defined by

(2.2) $$w_k(x;m) = m^{-1}\sum_{j=1}^{m} w(r^j x)r^{m-kj}.$$

It follows that

$$w_k(x;m) = \frac{1}{m}(r^{m-k}w(rx) + r^{m-2k}w(r^2x) + \cdots + r^{m-mk}w(r^m x))$$

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$$= \frac{1}{m}(r^{m-k}(w_0 + w_1rx + w_2r^2x^2 + \cdots) + r^{m-2k}(w_0 + w_1r^2x + w_2r^4x^2 + \cdots) + \cdots + r^{m-mk}(w_0 + w_1r^mx + w_2r^{2m}x^2 + \cdots))$$

$$= \frac{1}{m}\left(w_0\sum_{j=1}^{m} r^{m-jk} + w_1x\sum_{j=1}^{m} r^{m-jk+j} + \cdots + w_kx^k\sum_{j=1}^{m} r^{m-jk+jk} + \cdots\right)$$

$$= \frac{1}{m}\left(w_0\frac{r^{mk}-1}{r^k-1} + w_1x\,\frac{r^{m(k-1)}-1}{r^{k-1}-1} + \cdots + w_kx^kmr^m + \cdots\right)$$

$$= w_kx^k + w_{k+2m}x^{k+2m} + \cdots$$

$$= \sum_{j=0}^{\infty} w_{k+jm}x^{k+jm} \tag{i}$$

$$= \sum_{j=0}^{\infty} (A(\alpha x)^{k+jm} + B(\beta x)^{k+jm})$$

$$= A\alpha^kx^k(1-\alpha^mx^m)^{-1} + B\beta^kx^k(1-\beta^mx^m)^{-1}$$

$$= x^k(w_k - q^mw_{k-m}x^m)(1 - V_mx^m + q^mx^{2m})^{-1}. \tag{ii}$$

Hence, by cancelling the common factor x^k and replacing x^m by x, we get from the lines (i) and (ii)

$$(1 - V_mx + q^mx^2)\sum_{j=0}^{\infty} w_{k+jm}x^j = w_k - q^mw_{k-m}x.$$

We then equate the coefficients of x^j to get the lacunary recurrence relation for $\{w_n\}$:

$$w_{k+mj} - V_mw_{k+m(j-1)} + q^mw_{k+m(j-2)} = (w_k - V_mw_{k-m} + q^mw_{k-2m})\delta_{j0}, \tag{2.3}$$

where δ_{nm} is the Kronecker delta:

$$\delta_{nm} = 1\ (n = m),\quad \delta_{nm} = 0\ (n \neq m).$$

When j is zero, we get the trivial case $w_k = w_k$. When j is unity, we get

$$w_{k+m} - V_mw_k + q^mw_{k-m} = 0,$$

which is equation (3.16) of Horadam [5]. It is of interest to rewrite (2.3) as

$$w_{nm} = V_nw_{n(m-1)} + q^nw_{n(m-2)}\quad (m \geq 2,\ n \geq 1). \tag{2.4}$$

Thus $\quad w_{2n} = V_nw_n + aq^n,$

and $\quad w_{3n} = V_nw_{2n} + q^nw_n.$

The recurrence relations (2.3) and (2.4) are called lacunary because there are gaps in them. For instance, there are missing numbers between $w_{n(m-1)}$ and w_{nm} in (2.4); when $m = 2$ and $n = 3$, (2.4) becomes

$$w_6 = V_3w_3 + aq^3,$$

and the missing numbers are w_4 and w_5. A general solution of (2.4), in terms of w_n, is

(2.5) $$w_{mn} = U_m(V_n, -q)w_n + aU_{m-1}(V_n, -q)q^n.$$

The proof follows by induction on m. For $m = 2$ from (1.1) and (1.2),

$$U_2(V_n, -q) = V_n \quad \text{and} \quad U_1(V_n, -q) = 1.$$

If we assume (2.5) is true for $m = 3, 4, \ldots, r - 1$, then from (2.4)

$$\begin{aligned} w_{rn} &= V_n w_{n(r-1)} + q^n w_{n(r-2)} \\ &= V_n U_{r-1}(V_n, -q)w_n + aV_n U_{r-2}(V_n, -q)q^n \\ &\qquad + q^n U_{r-2}(V_n, -q)w_n + aq^n U_{r-3}(V_n, -q)q^n \\ &= (V_n U_{r-1}(V_n, -q) + q^n U_{r-2}(V_n, -q))w_n \\ &\qquad + a(V_n U_{r-2}(V_n, -q) + q^n U_{r-3}(V_n, -q))q^n \\ &= U_r(V_n, -q)w_n + aU_{r-1}(V_n, -q)q^n. \end{aligned}$$

3. FUNCTIONAL RECURRENCE RELATION

Following Carlitz [1], we define

(3.1) $$w_n^*(x) = w_n^*(x, \lambda) = \sum_{k=0}^{\infty} w_{n+k}\binom{x}{k}\lambda^k.$$

Then, $w_n^*(0) = w_n$, and

(3.2) $$\begin{aligned} w_{n+1}^*(x) &= \sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k}\lambda^k \\ &= \sum_{k=0}^{\infty} (pw_{n+k} - qw_{n+k+1})\binom{x}{k}\lambda^k \\ &= pw_n^*(x) - qw_{n-1}^*(x), \end{aligned}$$

which is a second-order functional recurrence relation. Moreover, we can show that the power series in (3.1) converges for a sufficiently small λ as follows:

$$\begin{aligned} w_n^*(x+1) - w_n^*(x) &= \sum_{k=0}^{\infty} w_{n+k}\left\{\binom{x+1}{k} - \binom{x}{k}\right\}\lambda^k \\ &= \lambda\sum_{k=1}^{\infty} w_{n+k}\binom{x}{k-1}\lambda^{k-1} \\ &= \lambda\sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k}\lambda^k \\ &= \lambda w_{n+1}^*(x). \end{aligned}$$

If we use $w_n = A\alpha^n + B\beta^n$, where

$$A = \frac{b - a\beta}{\alpha - \beta} \quad \text{and} \quad B = \frac{a\alpha - b}{\alpha - \beta},$$

then we get that

$$w_n^*(x) = \sum_{k=0}^{\infty}\left\{A\alpha^n\binom{x}{k}(\alpha\lambda)^k + B\beta^n\binom{x}{k}(\beta\lambda)^k\right\}$$

$$= A\alpha^n(1 + \lambda\alpha)^x + B\beta^n(1 + \lambda\beta)^x.$$

It follows that

$$w_n^*(x + y) = A\alpha^n(1 + \lambda\alpha)^{x+y} + B\beta^n(1 + \lambda\beta)^{x+y}$$

$$= \sum_{k=0}^{\infty}\left\{A\alpha^{n+k}(1 + \lambda\alpha)^x + B\beta(1 + \lambda\beta)^x\right\}\binom{y}{k}\lambda^k$$

$$= \sum_{k=0}^{\infty} w_{n+k}^*(x)\binom{y}{k}\lambda^k.$$

Similarly, we have for $E = pab - qa^2 - b^2$, and $E_w = 1 + p\lambda + q\lambda^2$:

$$\begin{aligned}
&w_{n-1}^*(x)w_{n+1}^*(x) - w_n^{*2}(x)\\
&= \left\{A\alpha^{n-1}(1 + \lambda\alpha)^x + B\beta^{n-1}(1 + \lambda\beta)^x\right\}\left\{A\alpha^{n+1}(1 + \lambda\alpha)^x + B\beta^{n+1}(1 + \lambda\beta)^x\right\}\\
&\qquad - \left\{A\alpha^n(1 + \lambda\alpha)^x + B\beta^n(1 + \lambda\beta)^x\right\}\\
&= Ed^{-2}(\alpha^{n-1}\beta^{n+1} - 2\alpha^n\beta^n + \alpha^{n+1}\beta^{n-1})((1 + \lambda\alpha)(1 + \lambda\beta))^x\\
&= q^{n-1}Ed^{-2}(\beta^2 - 2\alpha\beta + \alpha^2)E_w^x\\
&= q^{n-1}EE_w^x,
\end{aligned}$$

which is a generalization of equation (4.3) of Horadam [5]:

$$w_{n-1}w_{n+1} - w_n^2 = q^{n-1}E.$$

The same type of approach yields

$$aw_{m+n}^*(x + y) + (b - pq)w_{m+n-1}^*(x + y) = w_m^*(x)w_n^*(y) - qw_{m-1}^*(x)w_{n-1}^*(y)$$

as a generalization of Horadam's equation (4.1) [5]:

$$aw_{m+n} + (b - pq)w_{m+n-1} = w_m w_n - qw_{m-1}w_{n-1}.$$

4. q-SERIES RECURRENCE RELATION

q-series are defined by

(4.1) $$(q)_n = (1 - q)(1 - q^2)\cdots(1 - q^n),\quad (q_0) = 1.$$

Arising out of these are the so-called q-binomial coefficients:

(4.2) $$\begin{bmatrix} n \\ k \end{bmatrix}_q = (q)_n/(q)_k(q)_{n-k}.$$

When q is unity, these reduce to the ordinary binomial coefficients. It also follows from (4.1) and (4.2) that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\beta/\alpha} = \frac{(1 - (\beta/\alpha)^n) \cdots (1 - (\beta/\alpha)^{n-k+1})}{(1 - \beta/\alpha)(1 - (\beta/\alpha)^2) \cdots (1 - (\beta/\alpha)^k)}$$

$$= \alpha^{k(n-k)} \frac{u_{n-1}u_{n-2} \cdots u_{n-k}}{u_0 u_1 \cdots u_{k-1}}$$

$$= U_n c_{nk} \alpha^{k(n-k)},$$

$$(4.3) \qquad C_{nk} = \frac{u_{n-2}u_{n-3} \cdots u_{n-k}}{u_0 u_1 \cdots u_{k-1}}.$$

Horadam [5] has shown that

$$w_{n+r} = w_n u_r - q w_{r-1} u_{n-1}.$$

Thus

$$w_{n+r} = \frac{w_r \alpha^{k(k-n-1)}}{C_{n-1,k}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{\beta/\alpha} - \frac{q w_{r-1} \alpha^{k(k-n)}}{C_{nk}} \begin{bmatrix} n \\ k \end{bmatrix}_{\beta/\alpha}$$

which yields

$$(4.5) \quad C_{n-1,k} C_{nk} w_{n-r} = \alpha^{k(k-n-1)} C_{nk} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{\beta/\alpha} w_r - q\alpha^{k(k-n)} C_{n-1,k} \begin{bmatrix} n \\ k \end{bmatrix}_{\beta/\alpha} w_{r-1}.$$

5. CONCLUSION

The q-series analogue of the binomial coefficient was studied by Gauss, and later developed by Cayley. Carlitz has used the q-series in numerous papers. Fairly clearly, other results for w_n could be obtained with it just as other properties of the functional recurrence relation for w_n could be readily produced.

The process of multisection of series is quite an old one, and the interested reader is referred to Riordan [11]. Lehmer [9] discusses lacunary recurrence relations.

C_{nk} was introduced by Hoggatt [3], who used the symbol C. Curiously enough, Gould [2] also used the symbol 'C' in his generalization of Bernoulli and Euler numbers. Gould's $C = b/a$ (a,b the roots of $x^2 - x - 1 = 0$) is related to Hoggatt's $C \equiv C_{nk}$ when $p = -q = 1$ by

$$(5.1) \qquad C = b \lim_{k \to \infty} (C_{k+1,k+1}/C_{kk}).$$

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ON SOME EXTENSIONS OF THE WANG-CARLITZ IDENTITY

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ABSTRACT

Two theorems are presented which generalize a recent Wang [6]-Carlitz [1] result. In addition, we also obtain its Abel analogue. The method of proof is dependent upon some of our recent work [2].

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Wang [6] proved the expansion

$$\sum_{k=1}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}(i_m+1)=\binom{n+2r+1}{2r+1}. \tag{1.1}$$

Recently, Carlitz [1] extended (1.1) to

$$\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\binom{i_m+a}{i_m}=\binom{n+ar+r+a}{n}. \tag{1.2}$$

Theorems 1 and 2 in this paper treat a number of different generalizations of (1.2). In particular, a special case of Theorem 1 gives the new expression:

$$\begin{aligned}&\sum_{k=0}^{r+1}\binom{r+1}{k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_j>0}}\prod_{m=1}^{k}\frac{(a+1)}{(a+1+ti_m)}\binom{a+ti_m+i_m}{i_m}\\ &=\frac{(a+1)(r+1)}{(a+1)(r+1)+tn}\binom{ar+r+a+tn+n}{n}.\end{aligned} \tag{1.3}$$

Letting $t=0$ in (1.3) yields (1.2).