A PROPERTY OF QUASI-ORTHOGONAL POLYNOMIALS

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We define the simple set of polynomials $\phi_n(x)$ to be quasi-orthogonal if

$$(\phi_n, \phi_k) = \int_a^b w(x)\phi_n(x)\phi_k(x)dx = \begin{cases} A_n & \text{if } k = n - 1 \\ B_n & \text{if } k = n \\ C_n & \text{if } k = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

We shall require A_n and C_n to be nonvanishing. It is to be noted that the $\phi_n(x)$ may or may not be orthogonal over some other combination of range $[a, b]$ and weighting function $w(x)$. Consider, for example, if the range is $[-1, 1]$, $w(x) = 1 + x$, and $\phi_n(x) = P_n(x)$, the Legendre Polynomial,

$$\int_{-1}^{1}(1 + x)P_n(x)P_m(x)dx = \begin{cases} \dfrac{2n}{(2n-1)(2n+1)} & \text{if } m = n - 1 \\ \dfrac{2}{2n+1} & \text{if } m = n \\ \dfrac{2(n+1)}{(2n+1)(2n+3)} & \text{if } m = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, P_n is quasi-orthogonal, but, of course, if $w(x) = 1$, P_n is also orthogonal.

However, the simple set

$$\psi_n = (2n + 1)P_n + P_{n-1}$$

is quasi-orthogonal, but it is not orthogonal with respect to any range and weighting function. This is easily illustrated by noting that:

$$x\psi_3 = \frac{4}{9}\psi_4 + \frac{1}{45}\psi_3 + \frac{403}{15(45)}\psi_2 - \frac{133}{(45)^2}\psi_1 - \frac{281}{90(45)}\psi_0.$$

Since the ψ_n do not satisfy a three-term recursion formula, they, by the converse of Favard's Theorem, are not an orthogonal set, no matter what $w(x)$ or $[a, b]$ is selected. Favard's Theorem and converse are as follows.

Theorem: If the $\psi_n(x)$ are a set of simple polynomials which satisfy a three-term recursion formula, $x\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}$, then the ψ_n are orthogonal with respect to some weighting function $w(x)$ and some range $[a, b]$ if the integration be considered in the Stieltjes sense.

Converse: If the ψ_n are a simple set of polynomials orthogonal with respect to a weighting function $w(x)$ and some range $[a, b]$, then the ψ_n satisfy the three-term recursion formula:

$$x\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}.$$

For quasi-orthogonal polynomials, the following property will be satisfied:

Theorem: If $R_n(x)$ is a set of simple quasi-orthogonal polynomials over $[a, b]$ with respect to $w(x)$, then the necessary and sufficient condition that $R_n(x)$ also be orthogonal over some range $[c, d]$ with respect to some weighting function $w_1(x)$ is given by the expression:

$$xR_{n-1} = \sum_{k=0}^{n} c_k R_k, \quad n \geq 2,$$

$C_0 \neq 0$ if $n = 2$ and $C_0 = 0$ if $n \geq 3$.

Proof: The quasi-orthogonal character of R_n leads at once to the set of equations:

$$\begin{aligned}
(xR_{n-1}, R_{n+1}) &= C_n(R_n, R_{n+1}) \\
(xR_{n-1}, R_n) &= C_{n-1}(R_{n-1}, R_n) + C_n(R_n, R_n) \\
(xR_{n-1}, R_{n-1}) &= C_{n-2}(R_{n-2}, R_{n-1}) + C_{n-1}(R_{n-1}, R_{n-1}) + C\;(R\;, R_{n-1}) \\
(xR_{n-1}, R_{n-2}) &= C_{n-3}(R_{n-3}, R_{n-2}) + C_{n-2}(R_{n-2}, R_{n-2}) + C_{n-1}(R_{n-1}, R_{n-2}) \\
(xR_{n-1}, R_{n-3}) &= C_{n-4}(R_{n-4}, R_{n-3}) + C_{n-3}(R_{n-3}, R_{n-3}) + C_{n-2}(R_{n-2}, R_{n-3}) \\
0 &= C_{n-5}(R_{n-5}, R_{n-4}) + C_{n-4}(R_{n-4}, R_{n-4}) + C_{n-3}(R_{n-3}, R_{n-4}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= C_0(R_0, R_1) + C_1(R_1, R_1) + C_2(R_2, R_1) \\
0 &= C_0(R_0, R_0) + C_1(R_1, R_0).
\end{aligned}$$

The zero terms on the right occur, since

$$(xR_{n-1}, R_{n-4}) = (R_{n-1}, xR_{n-4}) = \left(R_{n-1}, \sum_{k=0}^{n-2} b_k R_k\right)$$

and

$$(R_{n-1}, R_k) = 0 \text{ for } k \leq n - 3.$$

We begin at the bottom of the chain and observe that if $C_0 = 0$, C_1 is also 0. Then the penultimate equation yields $C_2 = 0$. Continuing,

$$C_0 = C_1 = \cdots = C_{n-3} = 0.$$

To show $C_{n-2} \neq 0$, note that when $C_{n-2} = 0$, the fifth equation of the chain requires:

$$0 = (xR_{n-1}, R_{n-3}) = (R_{n-1}, xR_{n-3}) = \left(R_{n-1}, \sum_{k=0}^{n-2} a_k R_k\right) = a_{n-2}(R_{n-1}, R_{n-2}).$$

Now, $a_{k-2} \neq 0$, since from the equation

$$xR_{n-3} = \sum_{k=0}^{n-2} a_k R_k$$

we see that $a_{n-2} = \dfrac{h_{n-3}}{h_{n-2}}$, where h_{n-3} is the coefficient of x^{n-3} and h_{n-2} is the coefficient of x^{n-2} in R_{n-2}. Since the R_n are a simple set of polynomials, these cannot vanish. Therefore,

$$C_{n-2} \neq 0.$$

So, $C_0 = 0$ implies

$$xR_{n-1} = C_n R_n + C_{n-1}R_{n-1} + C_{n-2}R_{n-2}.$$

Hence, by Favard's Theorem, these R_n must be an orthogonal set with respect to some weighting function $w_1(x)$ and some range $[c, d]$ if the integral be considered a Stieltjes integral.

If $C_0 \neq 0$, the R_n do not satisfy a three-term recursion formula (unless $n = 2$) and by applying the contrapositive of the converse, we see that the R_n cannot be an orthogonal set with respect to any weighting function and range.

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ON SOME SYSTEMS OF DIOPHANTINE EQUATIONS INCLUDING THE ALGEBRAIC SUM OF TRIANGULAR NUMBERS

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The natural number of the form

$$t_n = \binom{n+1}{2} = \frac{1}{2}n(n+1),$$

where n is a natural number, is referred to as the nth triangular number. The aim of this work is to give solutions of some equations and systems of equations in triangular numbers.

1. THE EQUATION $t_{t_x} + t_{t_y} = t_{t_z}$

It is well known that the equation

$$t_x + t_y = t_z \tag{1}$$

has infinitely many solutions in triangular numbers t_x, t_y, and t_z. For example, it follows immediately from the formula:

$$t_{(2n+1)k} + t_{4t_n k+n} = t_{(4t_n+1)k+n}. \tag{2}$$

We can ask whether there exists a solution of the equation:

$$t_{t_x} + t_{t_y} = t_{t_z}. \tag{3}$$

The answer to this question is positive, because there exist two solutions:

$$t_{t_{59}} + t_{t_{77}} = t_{t_{83}} \quad \text{and} \quad t_{t_{104}} + t_{t_{213}} = t_{t_{216}}.$$