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GENERATING FUNCTIONS FOR RECURRENCE RELATIONS

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1. INTRODUCTION

In a previous paper [3] the author gave explicit solutions for four recurrence relations. The first was a basic relation with special initial conditions. The solution was shown to be related to the decompositions of the integer n relative to the first m positive integers. The second basic relation then restricted the first so that the solution was related to the decomposition of n relative to a subset of the first m positive integers. Then the initial conditions for both were extended to any arbitrary values.

In the next section we shall give the generating functions for all four of these cases, starting with the initial condition of highest index. We also note the form of the function for arbitrary indices for the initial conditions. Finally, we give a second function that generates all the initial conditions.

In Section 3 we give a simple example of the fourth kind of relation. We determine the first few terms of this relation and then compute its generating function. Then we consider relations given in [1] and [2] and determine their generating functions.

2. THE BASIC GENERATING FUNCTION

We shall consider a recurrence relation defined by

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\; G_{1-m},\; \ldots,\; G_0 \text{ arbitrary.}$$

For notation, we shall refer to its generating function as $R_m(G; x)$. The first term generated will be G_0. Later, we shall give a second function that will start with G_{1-m}.

Theorem 2.1: The generating function for the recurrence relation G_n is as follows:

$$R_m(G;\; x) = \left(G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

To prove that this does generate G_n, we set this equal to $\sum_{n=0}^{\infty} G_n x^n$ and then multiply by the factor with the negative exponent. This gives

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_s x^s - \sum_{s=1}^{m} \sum_{n=0}^{\infty} r_s G_n x^{s+n}.$$

In the last summation, we replace n by $n - s$ and transpose it to the left side so that

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n + \sum_{s=1}^{m} \sum_{n=s}^{\infty} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

We now break the second sum at $n = m$ and then interchange the orders of summation. We have

$$G_0 + \sum_{n=1}^{m-1} \sum_{s=n+1}^{m} r_s G_{n-s} x^n + \sum_{n=1}^{m} \sum_{s=1}^{n} r_s G_{n-s} x^n + \sum_{n=m+1}^{\infty} \sum_{s=1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

Note that for the first sum, if $n = m$, there would be no second sum, so we can combine the first three summations to give

$$G_0 + \sum_{n=1}^{\infty} \sum_{s=1}^{m} r_s G_{n-s} x^n = \sum_{n=0}^{\infty} G_n x^n.$$

It remains only to observe that the inner sum on the left is just G_n, so we have the desired result.

We now specialize this result for the U_n relation.

Corollary 2.2: The generating function for the relation

$$U_t = \sum_{s=1}^{m} r_s U_{t-s}; \; U_{1-m} = \cdots = U_{-1} = 0, \; U_0 = 1,$$

is given by

$$R_m(U;\, x) = \left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

In Theorem 2.1 the double summation of the numerator is zero since all initial conditions involved are zero. The other initial condition is 1, so the first factor is 1.

An implication of this result is that the generating function for G_n is obtained from that of U_n by multiplication by a polynomial of degree $m - 1$.

In [3] we generalized both the U_n and the G_n relations to the V_n and the H_n relations. This was accomplished by taking a subset A of the integers from 1 to m, including m. The solutions then were obtained by replacing r_i with 0 if $i \notin A$. We shall do that for their generating functions.

Corollary 2.3: The generating function for the relation

$$V_t = \sum_{s \in A} r_s V_{t-s}; \; V_{1-m} = \cdots = V_{-1} = 0, \; V_0 = 1,$$

is given by

$$R_A(V;\, x) = \left(1 - \sum_{s \in A} r_s x^s\right)^{-1}.$$

This follows directly from Corollary 2.2 by replacing r_i with 0 if $i \notin A$.

The most general recurrence relation is the H_n. Its generating function is given in the next corollary.

Corollary 2.4: The recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s};\ H_{1-m},\ \ldots,\ H_0 \text{ arbitrary},$$

is given by

$$R_A(H;\ x) = \left(H_0 + \sum_{s \in A'} \sum_{n=1}^{s-1} r_s H_{n-s} x^n\right)\left(1 - \sum_{s \in A} r_s x^s\right)^{-1},$$

where A' is A with 1 deleted if $1 \in A$; otherwise $A' = A$.

For the proof of this, we first need to interchange the order of summation in the numerator of the function of Theorem 2.1. Then we replace r_i with 0 for $i \neq A$.

The theorem together with the three corollaries start generating the given relation with the initial condition of highest order. In all our cases, this was the one with index 0. We can modify the notation to obtain a generating function with any indices for the initial conditions.

Theorem 2.5: The recurrence relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1+p},\ \ldots,\ G_{m+p} \text{ arbitrary},$$

has for its generating function

$$\left(G_{m+p} x^{m+p} + \sum_{n=m+1+p}^{2m+1+p} \sum_{s=n-m+1}^{m} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

This reduces to Theorem 2.1 when $p = -m$, as can be verified.

The only change we have for the U_n and V_n relations is to have as the numerator $U_{m+p} x^{m+p}$ and $V_{m+p} x^{m+p}$, respectively. The change for the H_n relation is given in the next corollary.

Corollary 2.6: The recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s};\ H_{1-p},\ \ldots,\ H_{m+p} \text{ arbitrary},$$

has for its generating function

$$\left(H_{m+p} x^{m+p} + \sum_{s \in A'} \sum_{n=m+p+1}^{m+p+s-1} r_s H_{n-s} x^n\right)\left(1 - \sum_{s \in A} r_s x^s\right)^{-1}.$$

Once more, this reduces to the result of Corollary 2.4 for $p = -m$.

If it were desired to generate all the initial conditions, the generating function is given in the next theorem.

Theorem 2.7: A generating function for the relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s};\ G_{1+p},\ \ldots,\ G_{m+p} \text{ arbitrary},$$

is given by

$$\left(\sum_{n=1+p}^{m+p} G_n x^n - \sum_{n=2+p}^{m+p} \sum_{s=1}^{n-1-p} r_s G_{n-s} x^n\right)\left(1 - \sum_{s=1}^{m} r_s x^s\right)^{-1}.$$

If we set this equal to $\sum_{n=1+p}^{\infty} G_n x^n$ and clear the negative exponent, we have

$$\sum_{n=1+p}^{m+p} G_n x^n - \sum_{n=2+p}^{m+p} \sum_{s=1}^{n-1-p} r_s G_{n-s} x^n = \sum_{n=1+p}^{\infty} G_n x^n - \sum_{s=1}^{m} \sum_{n=1+p}^{\infty} r_s G_n x^{n+s}.$$

To simplify this expression, we use the first term on the left to reduce the first term on the right. We transpose the second sum on the right. Further, we change the summation on n by replacing n by $n - s$, and then break the sum at $n = m + p$. This gives

$$\left(\sum_{s=1}^{m}\sum_{n=1+s+p}^{m+p} r_s G_{n-s}x^n - \sum_{n=2+p}^{m+p}\sum_{s=1}^{n-1+p} r_s G_{n-s}x^n\right) + \sum_{s=1}^{m}\sum_{n=m+1+p}^{\infty} r_s G_{n-s}x^n = \sum_{n=m+1+p}^{\infty} G_n x^n.$$

If $s = m$ in the first sum, we would have no second sum; thus we need sum only to $m - 1$. It can be verified that these two summations are the same. Finally, interchanging the summation on the last term on the left will give the right side from the definition of the G_n relation.

For the U_n and V_n relations, this gives the same generating function we had before.

3. EXAMPLES OF THE GENERATING FUNCTIONS

A simple example of an H_n relation will illustrate the results of the last section. Let $A = \{2, 5\}$ so $m = 5$ and $H_t = r_2H_{t-2} + r_5H_{t-5}$ with H_{-4}, H_{-3}, H_{-2}, H_{-1}, H_0 all arbitrary. It can be readily verified that the application of the definition of the relation yields, for the first seven terms,

$$\begin{aligned}
H_1 &= r_2H_{-1} + r_5H_{-4}\\
H_2 &= r_2H_0 + r_5H_{-3}\\
H_3 &= r_2^2H_{-1} + r_2r_5H_{-4} + r_5H_{-2}\\
H_4 &= r_2^2H_0 + r_2r_5H_{-3} + r_5H_{-1}\\
H_5 &= r_2^3H_{-1} + r_2^2r_5H_{-4} + r_2r_5H_{-2} + r_5H_0\\
H_6 &= r_2^3H_0 + r_2^2r_5H_{-3} + 2r_2r_5H_{-1} + r_5^2H_{-4}\\
H_7 &= r_2^4H_{-1} + r_2^3r_5H_{-4} + r_2^2r_5H_{-2} + 2r_2r_5H_0 + r_5^2H_{-3}.
\end{aligned}$$

The generating function is given by

$$(H_0 + r_2H_{-1}x + r_5(H_{-4}x + H_{-3}x^{-2} + H_{-2}x^3 + H_{-1}x^4))(1 - r_2x^2 - r_5x^5)^{-1}.$$

For the corresponding V_n relation, we have

$$V_1 = 0,\ V_2 = r_2,\ V_3 = 0,\ V_4 = r_2^2,\ V_5 = r_5,\ V_6 = r_2^3,\ V_7 = 2r_2r_5.$$

The generating function that gives all the initial conditions has for its numerator

$$H_{-4}x^{-4} + H_{-3}x^{-3} + (H_{-2} - r_2H_{-4})x^{-2} + (H_{-1} - r_2H_{-3})x^{-1} + (H_0 - r_2H_{-2}).$$

We shall list the five relations given in [2] and the one in [1, p. 4], and note their generating functions below them.

1. $G_k = rG_{k-1} + sG_{k-2}$; $G_0 = 0$, $G_1 = 1$

$$x(1 - rx - sx^2)^{-1} = x + rx^2 + (r^2 + s)x^3 \ldots$$

2. $F_k = F_{k-1} + F_{k-2}$; $F_0 = 0$, $F_1 = 1$

$$x(1 - x - x^2)^{-1} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$$

(This is the famous Fibonacci sequence.)

3. $M_k = rM_{k-1} + sM_{k-2}$; $M_0 = 2$, $M_1 = r$

$$(rx + 2sx^2)(1 - rx - sx^2)^{-1} = r + (r^2 + 2s)x^2 + \cdots$$

or

$$(2 - rx)(1 - rx - sx^2)^{-1} = 2 + rx + (r^2 + 2s)x^2 + \cdots$$

4. $L_k = L_{k-1} + L_{k-2}$; $L_0 = 2$, $L_1 = 1$

$$(x + 2x^2)(1 - x - x^2)^{-1} = x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

or

$$(2 - x)(1 - x - x^2)^{-1} = 2 + x + 3x^2 + 4x^3 + 7x^4 + \cdots$$

(This is the Lucas sequence.)

5. $U_k = rU_{k-1} + sU_{k-2}$; U_0, U_1 arbitrary

$$(U_1x + U_0sx^2)(1 - rx - sx^2)^{-1} = U_1x + (rU_1 + sU_0)x^2 + \cdots$$

or

$$(U_0 + (U_1 - U_0)x)(1 - rx - sx^2)^{-1} = U_0 + U_1x + (rU_1 + sU_0)x^2 + \cdots$$

6. $T_n = rT_{n-1} + sT_{n-2} - rsT_{n-3}$; T_0, T_1, T_2 arbitrary

$$(T_2x^2 + (sT_1 - rsT_0)x^3 - rsT_1x^4)(1 - rx - sx^2 + rsx^3)^{-1}$$
$$= T_2x^2 + (rT_2 + sT_1 - rsT_0)x^3 + \cdots$$

or

$$(T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2)(1 - rx - sx^2 + r2x^3)^{-1}$$
$$= T_0 + T_1x + T_2x^2 + (rT_2 + sT_1 - r\ T_0)x^3 + \cdots$$

From the solutions given in [2] and [1], it can be verified that we obtain the terms generated above.

The generating function given in Section 2 can be used to generate terms of any given recurrence relation. With specified values for the r_i and the initial conditions, the problem becomes a division of one polynomial by another.

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THE RESIDUES OF n^n MODULO p

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SUMMARY

In this paper we investigate the residues of $n^n \pmod{p}$, where $1 \leq n \leq p - 1$ and p is an odd prime. We find new upper bounds for the number of distinct residues of $n^n \pmod{p}$ that can occur. We also give lower bounds for the number of quadratic nonresidues and primitive roots modulo p that do not appear among the residues of $n^n \pmod{p}$. Further, we prove that given any arbitrarily large positive integer M, there exist sets of primes $\{p_i\}$ and $\{q_j\}$, both with positive density in the set of primes, such that the congruences

$$x^x \equiv 1 \pmod{p_i},\ 1 \leq x \leq p_i - 1 \tag{1}$$

and

$$x^x \equiv -1 \pmod{q_j},\ 1 \leq x \leq q_j - 1 \tag{2}$$

both have at least M solutions.