ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. Preference will be given to solutions that are submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-335 Proposed by Paul S. Bruckman, Concord, CA

Find the roots, in exact radicals, of the polynomial equation:

(1) $$p(x) = x^5 - 5x^3 + 5x - 1 = 0.$$

H-336 Proposed by Lawrence Somer, Washington, D.C.

Let p be an odd prime.

(a) Prove that if $p \equiv 3$ or $7 \pmod{20}$, then
$$5F^2_{(p-1)/2} \equiv -4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv -1 \pmod{p}.$$

(b) Prove that if $p \equiv 11$ or $19 \pmod{20}$, then
$$5F^2_{(p-1)/2} \equiv 4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv 1 \pmod{p}.$$

(c) Prove that if $p \equiv 13$ or $17 \pmod{20}$, then
$$F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 0 \pmod{p}.$$

(d) Prove that if $p \equiv 21$ or $29 \pmod{40}$, then
$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 1 \pmod{p}.$$

(e) Prove that if $p \equiv 1$ or $9 \pmod{40}$, then
$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv \pm 1 \pmod{p}.$$

Show that both the cases $F_{(p+1)/2} \equiv -1 \pmod{p}$ and $F_{(p+1)/2} \equiv 1 \pmod{p}$ do in fact occur.

H-337 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

(a) Evaluate the determinant:

$$\det A$$

$$\begin{array}{c} \begin{vmatrix} 1 & -4L_{2r} & 6L_{4r}+16 & -(4L_{6r}+24L_{2r}) & L_{8r}+16L_{4r}+36 \\ L_{2r} & -(3L_{4r}+10) & 3L_{6r}+25L_{2r} & -(L_{8r}+25L_{4r}+60) & 10L_{6r}+60L_{2r} \\ L_{4r} & -(2L_{6r}+6L_{2r}) & L_{8r}+12L_{4r}+30 & -(6L_{6r}+50L_{2r}) & 30L_{4r}+80 \\ L_{6r} & -(L_{8r}+7L_{4r}) & 7L_{6r}+21L_{2r} & -(21L_{4r}+70) & 70L_{2r} \\ L_{8r} & -8L_{6r} & 28L_{4r} & -56L_{2r} & 140 \end{vmatrix} \end{array} \begin{array}{c} (e) \\ (d) \\ (c) \\ (b) \\ (a) \end{array}$$

(b) Show that:

$$
\begin{aligned}
625F_{2r}^2 &= L_{8r}^2 - 8L_{6r}^2 + 28L_{4r}^2 - 56L_{2r}^2 + 140 \\
&= 8L_{6r}^2 + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r+10})^2 - 280L_{2r}^2 \\
&= 28L_{4r}^2 - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2 \\
&\qquad + 20(3L_{4r} + 8)^2 \\
&= -56L_{2r}^2 + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2 \\
&\qquad - 40(L_{6r} + 6L_{2r})^2.
\end{aligned}
$$

Grace Note: If the elements of this determinant are the coefficients of a 5 X 5 linear homogeneous system, then the solution to the 4 X 5 system represented by equations (b), (c), (d), (e) is given by the elements of the first column. The solution to (a), (c), (d), (e) is given by the elements of the second column; and so on.

H-338 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

An integer n is abundant if $\sigma(n) > 2n$, where $\sigma(n)$ is the sum of the divisors of n. Show that there is a probability of at least:

(a) 0.15 that a Fibonacci number is abundant;
(b) 0.10 that a Lucas number is abundant.

SOLUTIONS

Sum Enumerator

H-316 Proposed by B. R. Myers, Univ. of British Columbia, Vancouver, Canada (Vol. 18, no. 2, April 1980)

The enumerator of compositions with exactly k parts is $(x + x^2 + \cdots)^k$, so that

$$[W(x)]^k = (w_1x + w_2x^2 + \cdots)^k \tag{1}$$

is then the enumerator of weighted k-part compositions. After Hoggatt and Lind ["Compositions and Fibonacci Numbers," *The Fibonacci Quarterly* 7 (1969):253-66], the number of weighted compositions of n can be expressed in the form

$$C_n(w) = \sum_{\nu(n)} w_{a_1} \ldots w_{a_k} \qquad (n > 0), \tag{2}$$

where $w = \{w_1, w_2, \ldots\}$ and where the sum is over all compositions $a_1 + \cdots + a_k$ of n (k variable). In particular (*ibid.*),

$$\sum_{\nu(n)} a_1 \ldots a_k = F_{2n}(1, 1), \tag{3}$$

where $F_k(p, q)$ is the kth number in the Fibonacci sequence

$$
\begin{aligned}
F_1(p, q) &= p \qquad (\geq 0) \\
F_2(p, q) &= q \qquad (\geq p) \\
F_{n+2}(p, q) &= F_{n+1}(p, q) + F_n(p, q) \qquad (n \geq 1).
\end{aligned} \tag{4}
$$

Show that

$$\sum_{\nu(n)} (a_1 \pm 1)a_1 \ldots a_k = 2[F_{2n-1}(1, 1) - 1] \tag{5}$$

and, hence, that

$$\sum_{\nu(n)} (a_1 - 1)a_1 \ldots a_k + \sum_{\nu(n)} a_1 \ldots a_k = F_{2n}(1, 1 + 2m) - 2m \qquad (m \geq 0). \tag{6}$$

Solution by the proposer.

As in the example $C_4 = 1(3 + 21 + 12 + 111) + 2(2 + 11) + 3(1) + 4$, the compositions C_n of n are given by

(7) $$C_n = 1C_{n-1} + 2C_{n-2} + \cdots + nC_0,$$

where C_0 is the identity element ($nC_0 \equiv n$). Equation (7) implies that

(8) $$\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = \sum_{\nu(n-1)} 1^2a_1a_2 \ldots a_k + \sum_{\nu(n-2)} 2^2a_1a_2 \ldots a_k + \cdots + \sum_{\nu(1)} n^2a_1a_2 \ldots a_k,$$

so that, by (3),

(9) $$\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = n^2F_1(1, 1) + \sum_{i=1}^{n-1} i^2F_{2(n-i)}(1, 1).$$

It is not difficult to show (for example, from Problems P.36–P.37 on p. 9 of Bro. U. Alfred's "An Introduction to Fibonacci Discovery," The Fibonacci Association, 1965) that

(10) $$F_{2n}(1, 3) - 2 = n^2F_1(1, 1) + \sum_{i=1}^{n-1} i^2F_{2(n-i)}(1, 1),$$

so that, from (9) and (10)

(11) $$\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = F_{2n}(1, 3) - 2.$$

Equations (5) and (6) follow routinely from manipulation of (11) in conjunction with the identity

(12) $$F_k(1, 1 + 2m) = (m - 1)F_{k-1}(1, 1) + F_{k+1}(1, 1)$$

for $m \geq 0$, $k > 1$.

Prime Time

H-317 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 18, no. 3, April 1980)

Let $\{G_n\}_{n=0}^{\infty}$ be any generalized Fibonacci sequence such that $G_{n+2} = G_{n+1} + G_n$, $(G_0, G_1) = 1$, and $\{G_n\}$ is not a translation of the Fibonacci sequence. Show that there exists at least one prime p such that both

$$G_n + G_{n+1} \equiv G_{n+2} \pmod p \quad \text{and} \quad G_{n+1} \equiv rG_n \pmod p$$

for a fixed $r \not\equiv 0 \pmod p$ and for all $n \geq 0$.

Solution by Paul S. Bruckman, Concord, CA.

We define the discriminant of $\{G_n\}_{n=0}^{\infty}$ by

(1) $$D(G_n) \equiv G_n^2 - G_{n-1}G_{n+1}, \quad n = 1, 2, \ldots .$$

This satisfies the invariance relation

(2) $$D(G_n) = (-1)^{n-1}D(G_1).$$

Since $\{G_n\}_{n=0}^{\infty}$ is not a translation of the Fibonacci sequence, thus $|D(G_1)| > 1$. Hence $D(G_1)$ is divisible by a prime p. If we were to have $p|G_0$, then we would also have $p|G_1$, and conversely, since $D(G_1) = G_1^2 - G_0G_1 - G_0^2$. This, however, would contradict the condition $(G_0, G_1) = 1$; hence $p\nmid G_0$, $p\nmid G_1$.

The congruence

(3) $$G_n + G_{n+1} \equiv G_{n+2} \pmod p$$

is a trivial consequence of the recursion satisfied by $\{G_n\}_{n=0}^{\infty}$. If the congruence

(4) $$rG_n \equiv G_{n+1} \pmod{p}$$

is to hold for all $n \geq 0$, it must in particular hold for $n = 0$, for some fixed r, so that $rG_0 \equiv G_1 \pmod{p}$. Since $G_0 \not\equiv 0 \pmod{p}$, this uniquely determines $r \pmod{p}$:

(5) $$r \equiv G_1(G_0)^{-1} \pmod{p}, \text{ where } p \text{ is any prime divisor of } D(G_1).$$

Note that $r \not\equiv 0 \pmod{p}$. To show that this r satisfies (4) for all $n \geq 0$, we proceed by induction on n. Let S denote the set of nonnegative integers n such that (4) holds, where r is given by (5). Clearly, $0 \in S$. Also, $rG_1 \equiv G_1^2(G_0)^{-1} \equiv G_0G_2(G_0)^{-1} \equiv G_2 \pmod{p}$, which shows that $1 \in S$. Suppose $k \in S$, $k = 0, 1, \ldots, m$. Then $rG_{m+1} \equiv r(G_m + G_{m-1}) \equiv G_{m+1} + G_m \equiv G_{m+2} \pmod{p}$. Hence $k \in S \Rightarrow (k + 1) \in S$. By induction, (4) is proved.

Also solved by the proposer.

Canonical Möbius

H-318 Proposed by James Propp, Harvard College Cambridge, MA (Vol. 18, no. 3, April 1980)

Define the sequence operator M so that for any infinite sequence $\{u_i\}$,

$$M(u_n) = M(u_n) - \sum_{i|n} M(u_i)\mu\left(\frac{n}{i}\right),$$

where μ is the Möbius function. Let the "Möbinacci Sequence" S be defined so that $S_1 = 1$ and $S_n = M(S_n) + M(M(S_n))$ for $n > 1$. Find a formula for S_n in terms of the prime factorization of n.

<u>*Remarks:*</u> I've been unable to solve this problem, but some special cases were easier. Let p be a prime. $S(1) = 1$, $S(p) = 1$. For $a \geq 2$,

$$M(S_{p^a}) = S_{p^{a-1}} \quad \text{and} \quad M(M(S_{p^a})) = S_{p^{a-2}},$$

so that $S_{p^a} = S_{p^{a-1}} + S_{p^{a-2}}$. Solving this difference equation, we get

$$S_{p^a} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^a - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^a$$

I have found no explicit formula for the case $n = p^a q^b$, but if one holds b fixed and finds S_n in terms of a, the characteristic equations seem to have only the the roots $\emptyset$ and $-1/\emptyset$, where $\emptyset = \frac{1}{2}(1 + \sqrt{5})$.

Fibonacci Never More

H-319 Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 18, no. 3, April 1980)

If $F_n < x < F_{n+1} < y < F_{n+2}$, then $x + y$ is never a Fibonacci number.

Solution by M. J. DeLeon, Florida Atlantic Univ., Boca Raton, FL.

Assume that $F_n < x < F_{n+1} < y < F_{n+2}$. Since $F_n < x$ and $F_{n+1} < y$, $F_{n+2} = F_n + F_{n+1} < x + y$. Since $x < F_{n+1}$ and $y < F_{n+2}$, $x + y < F_{n+1} + F_{n+2} = F_{n+3}$. Therefore,

$$F_{n+2} < x + y < F_{n+3}.$$

Since $F_n < F_{n+1} < F_{n+2}$, $n \geq 0$. Since $n \geq 0$, there is no Fibonacci number between F_{n+2} and F_{n+3}. Therefore $x + y$ is not a Fibonacci number.

Also solved by P. Bruckman, R. Giuli, G. Lord, F. D. Parker, B. Prielipp, S. Singh, L. Somer, M. Wachtel, R. Whitney, and the proposer.

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