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A NOTE ON THE FIBONACCI SEQUENCE OF ORDER K AND THE MULTINOMIAL COEFFICIENTS

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In the sequel, k is a fixed integer greater than or equal to 2, and n is a nonnegative integer as specified. Recall the following definition [6]:

Definition

The sequence $\{f_n^{(k)}\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geqslant k+1. \end{cases}$$

Gabai [2] called $\{f_n^{(k)}\}_{n=-\infty}^{\infty}$ with $f_n^{(k)} = 0$ for $n \leqslant -1$ the Fibonacci k-sequence. See, also, [1], [4], and [5].

Recently, Philippou and Muwafi [6] obtained the following theorem, which provides a formula for the nth term of the Fibonacci sequence of order k in terms of the multinomial coefficients.

Theorem 1

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$.

Presently, a new proof of this theorem is given which is simpler and more direct. In addition, the following theorem is derived, which provides a new formula for the nth term of the Fibonacci sequence of order k in terms of the binomial coefficients.

Theorem 2

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{-(k+1)i} - 2^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{-(k+1)i}, \quad n \geq 1,$$

where, as usual, $[x]$ denotes the greatest integer in x.

The proofs of the above formulas are based on the following lemma.

Lemma

Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and denote its generating function by $g_k(x)$. Then, for $|x| < 1/2$,

$$g_k(x) = \frac{x - x^2}{1 - 2x + x^{k+1}} = \frac{x}{1 - x - x^2 - \cdots - x^k}.$$

Proof: We see from the definition that

$$f_2^{(k)} = 1,\ f_n^{(k)} - f_{n-1}^{(k)} = f_{n-1}^{(k)} \text{ for } 3 \leqslant n \leqslant k+1,$$

and

$$f_n^{(k)} - f_{n-1}^{(k)} = f_{n-1}^{(k)} - f_{n-1-k}^{(k)} \text{ for } n \geqslant k+1.$$

Therefore,

$$f_n^{(k)} = \begin{cases} 2^{n-2} & 2 \leqslant n \leqslant k \\ 2f_{n-1}^{(k)} - f_{n-1-k}^{(k)}, & n \geqslant k+1. \end{cases} \tag{1}$$

By induction on n, the above relation implies $f_n^{(k)} \leqslant 2^{n-2}$ $(n \geqslant 2)$ [5], which shows the convergence of $g_k(x)$ for $|x| < 1/2$. It follows that

$$g_k(x) = \sum_{n=0}^{\infty} x^n f_n^{(k)} = x + \sum_{n=2}^{k} x^n 2^{n-2} + \sum_{n=k+1}^{\infty} x^n f_n^{(k)}, \text{ by (1)}, \tag{2}$$

and

$$\begin{aligned} \sum_{n=k+1}^{\infty} x^n f_n^{(k)} &= 2 \sum_{n=k+1}^{\infty} x^n f_{n-1}^{(k)} - \sum_{n=k+1}^{\infty} x^n f_{n-1-k}^{(k)} \\ &= 2x\left(\sum_{n=0}^{\infty} x^n f_n^{(k)} - x - \sum_{n=2}^{k-1} x^n 2^{n-2}\right) - x^{k+1} g_k(x) \end{aligned} \tag{3}$$

$$= (2x - x^{k+1})g_k(x) - x^2 - \sum_{n=2}^{k} x^n 2^{n-2}.$$

The last two relations give $g_k(x) = x + (2x - x^{k+1})g_k(x) - x^2$, so that

$$g_k(x) = \frac{x - x^2}{1 - 2x + x^{k+1}} = \frac{x}{1 - x - x^2 - \cdots - x^k},$$

which shows the lemma.

Proof of Theorem 1: Let $|x| < 1/2$. Also let n_i $(1 \leqslant i \leqslant k)$ be non-negative integers as specified. Then

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = (1 - x - x^2 - \cdots - x^k)^{-1}, \text{ by the lemma,} \tag{4}$$

$$= \sum_{n=0}^{\infty} (x + x^2 + \cdots + x^k)^n, \text{ since } |x + x^2 + \cdots + x^k| < 1,$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \ \ldots, \ n_k} x^{n_1 + 2n_2 + \cdots + kn_k},$$

by the multinomial theorem. Now setting $n_i = m_i$ $(1 \leqslant i \leqslant k)$ and

$$n = m - \sum_{i=2}^{k} (i - 1)m_i,$$

we get

$$\sum_{n=0}^{\infty} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \ \ldots, \ n_k} x^{n_1 + 2n_2 + \cdots + kn_k} \tag{5}$$

$$= \sum_{m=0}^{\infty} x^m \sum_{\substack{m_1, \ldots, m_k \ni \\ m_1 + 2m_2 + \cdots + km_k = m}} \binom{m_1 + \cdots + m_k}{m_1, \ \ldots, \ m_k}.$$

Equations (4) and (5) imply

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = \sum_{n=0}^{\infty} x^n \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ \ldots, \ n_k}$$

from which the theorem follows.

Proof of Theorem 2: Set $S_k = \{x \in R;\ |x| < 1/2 \text{ and } |2x - x^{k+1}| < 1\}$, and let $x \in S_k$. Then

$$\sum_{n=0}^{\infty} x^n f_{n+1}^{(k)} = \frac{1-x}{1-2x+x^{k+1}}, \text{ by the lemma,} \tag{6}$$
$$= (1-x)\sum_{n=0}^{\infty}(2x-x^{k+1})^n$$
$$= (1-x)\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(-1)^i x^{n+ki},$$

by the binomial theorem. Now setting $i = j$ and $n = m - kj$, and defining the sequence $\{b_n^{(k)}\}_{n=0}^{\infty}$ by

$$b_n^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]}(-1)^i\binom{n-ki}{i}2^{-(k+1)i}, \; n \geqslant 0, \tag{7}$$

we get

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(-1)^i x^{n+ki} = \sum_{m=0}^{\infty}x^m \sum_{j=0}^{[m/(k+1)]}\binom{m-kj}{j}2^{m-(k+1)j}(-1)^j \tag{8}$$
$$= \sum_{m=0}^{\infty}x^m b_m^{(k)}.$$

Relations (6) and (8) give

$$\sum_{n=0}^{\infty}x^n f_{n+1}^{(k)} = (1-x)\sum_{n=0}^{\infty}x^n b_n^{(k)} = 1 + \sum_{n=1}^{\infty}x^n(b_n^{(k)} - b_{n-1}^{(k)}),$$

since $b_0^{(k)} = 1$ from (7). Therefore,

$$f_{n+1}^{(k)} = b_n^{(k)} - b_{n-1}^{(k)}, \; n \geqslant 1. \tag{9}$$

Relations (7) and (9) establish the theorem.

We note in ending that the above-mentioned same two relations imply

$$\sum_{i=1}^{n} f_i^{(k)} = 1 + \sum_{i=1}^{n-1}(b_n^{(k)} - b_{n-1}^{(k)}) = b_{n-1}^{(k)} \tag{10}$$
$$= 2^{n-1}\sum_{i=0}^{[(n-1)/(k+1)]}(-1)^i\binom{n-1-ki}{i}2^{-(k+1)i}, \; n \geqslant 1,$$

which reduces to

$$\sum_{i=1}^{n} F_i = 2^{n-1}\sum_{i=0}^{[(n-1)/3]}(-1)^i\binom{n-1-2i}{i}2^{-3i}, \; n \geqslant 1, \tag{11}$$

since $F_i = f_i^{(2)}$ $(i \geqslant 0)$ from the definition. Also, observing that

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \ (n \geqslant 1),$$

see, for example, Hoggatt [3, (I_1), p. 52], we get, from (11), the following identity for the Fibonacci sequence:

$$F_{n+2} = 1 + 2^{n-1} \sum_{i=0}^{[(n-1)/3]} (-1)^i \binom{n-1-2i}{i} 2^{-3i}, \ n \geqslant 1. \tag{12}$$

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