# HARMONIC, GEOMETRIC, AND ARITHMETIC MEANS IN GENERALIZED FIBONACCI SEQUENCES

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The Fibonacci sequence of numbers, $F_i$, can be defined as the sequence whose first two terms are unity and whose $n^{\text{th}}$ term (for $n > 2$) is equal to the sum of the $(n-1)^{\text{st}}$ term and the $(n-2)^{\text{nd}}$ term. The ratio of increase between successive terms rapidly approaches a constant value, the positive root of the equation

$$x^2 - x - 1 = 0, \tag{1}$$

which is $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618034$.

Fibonacci sequences can be generalized by increasing the number of previous terms that are summed to produce subsequent terms. Therefore, the Tribonacci sequence, $T_i$, has its $n^{\text{th}}$ term equal to the sum of its $(n-1)^{\text{st}}$, $(n-2)^{\text{nd}}$, and $(n-3)^{\text{rd}}$ terms. The ratio of increase between successive terms in the Tribonacci sequence is the real, positive root of the equation

$$x^3 - x^2 - x - 1 = 0, \tag{2}$$

which is $\tau = 1.839287$ (see [1], [2], and [3]).

In general, the $n$-bonacci sequence has its $i^{\text{th}}$ term equal to the sum of the previous $n$ terms. The ratio if increase is then the real, positive root of the equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0. \tag{3}$$

For $n \geqslant 2$, $\phi \leqslant x < 2$.

Because such generalized Fibonacci sequences soon approximate geometric sequences, all of the terms in those sequences (aside from initial values) are approximately equal to the geometric means of the immediately preceding and immediately following terms. At the same time, because of the Fibonacci nature of those sequences, each term is also approximately equal to the harmonic mean and exactly equal to the arithmetic mean of neighboring terms. Those properties are the focus of the present paper.

The harmonic, geometric, and arithmetic means of two positive numbers, $a$ and $b$, are defined as

$$HM(a, b) = \frac{2ab}{a + b} \tag{4}$$

$$GM(a, b) = \sqrt{ab}, \tag{5}$$

and

$$AM(a, b) = \frac{a + b}{2}, \tag{6}$$

respectively. They are related by the classical chain of inequalities

$$HM(a, b) \leqslant GM(a, b) \leqslant AM(a, b).$$

Now consider the question of finding a geometric sequence (i.e., a sequence of terms where the $i^{\text{th}}$ term is equal to the $(i-1)^{\text{st}}$ term times a constant) in which any four consecutive terms can be written in the form

$$a,\ HM(a, b),\ AM(a, b),\ b.$$

If we set $a = 1$ and denote the ratio by $x$, we must solve the set of equations

$$x = \frac{2b}{1 + b},\ x^2 = \frac{1 + b}{2},\ \text{and } x^3 = b, \tag{7}$$

which are consistent and reduce to

$$x^3 - 2x^2 + 1 = 0. \tag{8}$$

By inspection, $x = 1$ is a root of equation (8), indicating that a sequence of identical terms satisfies the stated conditions. To exclude that trivial solution, we divide equation (8) by $(x - 1)$ and find

$$x^2 - x - 1 = 0, \tag{9}$$

the equation for the ratio of the Fibonacci sequence. Thus, the integers in a Fibonacci sequence approximate the harmonic and arithmetic means of nearby Fibonacci numbers. Table 1 shows the first 15 Fibonacci numbers and indicates that the arithmetic mean relationship is exact from $n = 2$ onward, while the harmonic mean relationship is correct to within ±0.1 as early as $n = 6$.

TABLE 1

HARMONIC AND ARITHMETIC MEANS IN THE FIBONACCI SEQUENCE

| $n$ | Fibonacci Number $F_n$ | Harmonic Mean of $F_{n-1}$, $F_{n+2}$ | Arithmetic Mean of $F_{n-2}$, $F_{n+1}$ | Ratio $F_n/F_{n-1}$ |
|---|---|---|---|---|
| 1 | 1 | - | - | - |
| 2 | 1 | 1.500 | - | 1.000 |
| 3 | 2 | 1.667 | 2 | 2.000 |
| 4 | 3 | 3.200 | 3 | 1.500 |
| 5 | 5 | 4.875 | 5 | 1.667 |
| 6 | 8 | 8.077 | 8 | 1.600 |
| 7 | 13 | 12.952 | 13 | 1.625 |
| 8 | 21 | 21.029 | 21 | 1.615 |
| 9 | 34 | 33.982 | 34 | 1.619 |
| 10 | 55 | 55.011 | 55 | 1.618 |
| 11 | 89 | 88.993 | 89 | 1.618 |
| 12 | 144 | 144.004 | 144 | 1.618 |
| 13 | 233 | 232.997 | 233 | 1.618 |
| 14 | 377 | 377.002 | 377 | 1.618 |
| 15 | 610 | 609.999 | 610 | 1.618 |

The same approach can be applied to finding a geometric series where successive terms can be written in the form

$$a,\ HM(a, b),\ GM(a, b),\ AM(a, b),\ b.$$

With $a = 1$ and ratio $x$, the equations are

$$x = \frac{2b}{1 + b},\ x^2 = \sqrt{b},\ x^3 = \frac{1 + b}{2},\ \text{and } x^4 = b, \tag{10}$$

which are consistent and reduce to

$$x^4 - 2x^3 + 1 = 0. \tag{11}$$

If we again divide by $(x - 1)$ to eliminate the trivial solution, we have

$$x^3 - x^2 - x - 1 = 0, \tag{12}$$

the equation for the ratio of the Tribonacci sequence. Table 2 shows the first 15 Tribonacci numbers, and again the mean relationships emerge quite quickly.

TABLE 2

HARMONIC, GEOMETRIC, AND ARITHMETIC MEANS IN THE TRIBONACCI SEQUENCE

| $n$ | Tribonacci Number $T_n$ | Harmonic Mean of $T_{n-1}$, $T_{n+3}$ | Geometric Mean of $T_{n-2}$, $T_{n+2}$ | Arithmetic Mean of $T_{n-3}$, $T_{n+1}$ | Ratio $T_n/T_{n-1}$ |
|---|---|---|---|---|---|
| 1 | 1 | - | - | - | - |
| 2 | 1 | 1.750 | - | - | 1.000 |
| 3 | 2 | 1.857 | 2.646 | - | 2.000 |
| 4 | 4 | 3.692 | 3.606 | 4 | 2.000 |
| 5 | 7 | 7.333 | 6.928 | 7 | 1.750 |
| 6 | 13 | 12.886 | 13.266 | 13 | 1.857 |
| 7 | 24 | 23.914 | 23.812 | 24 | 1.846 |
| 8 | 44 | 44.134 | 44.011 | 44 | 1.833 |
| 9 | 81 | 80.934 | 81.093 | 81 | 1.841 |
| 10 | 149 | 148.982 | 148.916 | 149 | 1.840 |
| 11 | 274 | 274.051 | 274.020 | 274 | 1.839 |
| 12 | 504 | 503.967 | 504.029 | 504 | 1.839 |
| 13 | 927 | 927.000 | 926.965 | 927 | 1.839 |
| 14 | 1705 | 1705.018 | 1705.014 | 1705 | 1.839 |
| 15 | 3136 | 3135.985 | 3136.007 | 3136 | 1.839 |

Let us now formally generalize the above relationships between Fibonacci sequences and harmonic and geometric means.

Theorem

If positive, real numbers 1 and $b$ are the $m^{\text{th}}$ and $(m + n + 1)^{\text{st}}$ terms in a geometric sequence with ratio $x > 1$, and the $(m + 1)^{\text{st}}$ term is $HM(1, b)$ and the $(m + n)^{\text{th}}$ term is $AM(1, b)$, then the ratio of that geometric sequence is equal to the ratio of the corresponding $n$-bonacci sequence, i.e., the real, positive root of the equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0.$$

Proof: The terms in the geometric sequence must satisfy the equations

$$x = \frac{2b}{1 + b}, \quad x^n = \frac{1 + b}{2}, \quad x^{n+1} = b. \tag{13}$$

which are consistent and which reduce to

$$x^{n+1} - 2x^n + 1 = 0. \tag{14}$$

Eliminating the root $x = 1$ by dividing equation (14) by $(x - 1)$ yields the $n$-bonacci equation

$$x^n - x^{n-1} - \cdots - x - 1 = 0. \tag{15}$$

Thus, in any $n$-bonacci sequence $S$, the term $S_m$ is approximately equal to the harmonic mean of terms $S_{m-1}$ and $S_{m+n}$, and exactly equal to the arithmetic mean of terms $S_{m-n}$ and $S_{m+1}$.

One might ask whether all of the $n$ terms between $S_m$ and $S_{m+n+1}$ in an $n$-bonacci sequence can be expressed in terms of generalized means, where a generalized mean of order $t$, $M(t)$, is given by

$$M(t) = \left[\frac{1}{2}(a^t + b^t)\right]^{1/t}. \tag{16}$$

When $t = 1$ equation (16) yields the arithmetic mean, when $t = -1$ it yields the harmonic mean, and in the limit as $t$ goes to 0, it yields the geometric mean [4, p. 10]. The answer, in general, is no, as in shown in the following paragraph.

Let us examine the case where $n = 4$, i.e., the Tetranacci sequence. If six consecutive terms can be expressed in the form

$$a,\ HM(a,\ b),\ M(-t),\ M(t),\ AM(a,\ b),\ b,$$

then we have the equations (with $a = 1$):

$$x = \frac{2b}{1 + b},\ x^2 = \left[\frac{1}{2}(1 + b^{-t})\right]^{-1/t},\ x^3 = \left[\frac{1}{2}(1 + b^t)\right]^{1/t},\ x^4 = \frac{1 + b}{2},\ x^5 = b. \tag{17}$$

The first and fourth equations, for the harmonic and arithmetic means, reduce to

$$x^5 - 2x^4 + 1 = 0. \tag{18}$$

The second and third equations are consistent (because the values $-t$ and $t$ are used), and reduce to

$$x^{5t} - 2x^{3t} + 1 = 0. \tag{19}$$

Equations (18) and (19) are clearly inconsistent, however, as there is no value of $t$ that can simultaneously satisfy the conditions $5t = 5$ and $3t = 4$. Thus, aside from the trivial solution $x = 1$, it is not possible to represent four consecutive terms of a Tetranacci sequence as generalized means of the two adjacent terms.

In summary, harmonic and arithmetic means naturally arise in Fibonacci-type sequences. In the geometric series that forms the limit of every $n$-bonacci sequence, the $m^{\text{th}}$ term will be equal to the harmonic mean of the $(m - 1)^{\text{st}}$ and the $(m + n)^{\text{th}}$ terms and the arithmetic mean of the $(m - n)^{\text{th}}$ and $(m + 1)^{\text{st}}$ terms. The aesthetic appeal of Fibonacci proportions may be due, in part, to their natural blending of harmonic, geometric, and arithmetic means.

## REFERENCES


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