A CONGRUENCE FOR A CLASS OF EXPONENTIAL NUMBERS

A. KYRIAKOYSSIS
University of Athens, Couponia, Athens (621)-Greece
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1. INTRODUCTION

A sequence of exponential numbers, say P_n, is defined by its exponential generating function as

$$\sum_{n=0}^{\infty} P_n x^n/n! = \exp\{g(x)\}$$

for some (formal) power series $g(x)$ with constant term zero.

As regards Bell numbers $[g(x) = e^x - 1]$, Lunnon, Pleasants, and Stephens [6] showed that for each positive integer n, there exist integers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that, for all $m \geqslant 0$,

$$P_{m+n} + \alpha_{n-1}P_{m+n-1} + \cdots + \alpha_0 P_m \equiv 0 \pmod{n!}.$$

In this paper, we show a similar congruence for the exponential numbers P_n when $g(x)$ is a certain series function (Section 2). Special cases include numbers P_n equal to the number of permutations of n elements having cycles with given maximal and minimal size or equal to the sum of the horizontal entries of the table of Jordan [5, p. 223], also for P_n equal to the generalized derangement numbers.

2. THE CONGRUENCE

Theorem. Suppose

$$g(x) = \sum_{j=1}^{\infty} b_j \frac{x^j}{j}$$

where the b_j are integers. Let

$$e^{g(x)} = \sum_{n=0}^{\infty} P_n \frac{x^n}{n!} \tag{1}$$

and let

$$\frac{y^k}{k!} e^{-g(y)} = \sum_{n=k}^{\infty} D_{n,k} \frac{y^n}{n!}. \tag{2}$$

Then, for each $m, n \geqslant 0$,

$$\sum_{k=0}^{n} D_{n,k} P_{m+k} \equiv 0 \pmod{n!}.$$

Proof: Let $f(x) = e^{g(x)}$. Then

$$e^{-g(y)}f(x+y) = \sum_{k=0}^{\infty} e^{-g(y)}\frac{y^k}{k!}f^{(k)}(x) = \sum_{m,n=0}^{\infty}\frac{x^m}{m!}\frac{y^n}{n!}\sum_{k=0}^{n}D_{n,k}P_{m+k}.$$

Thus, it is sufficient to show that the coefficient of $x^m/m!$ in $e^{-g(y)}f(x+y)$ is a power series in y with integer coefficients.

Now we have

$$e^{-g(y)}f(x+y) = \exp\left[\sum_{i=1}^{\infty} g^{(i)}(y)\frac{x^i}{i!}\right].$$

Since $g'(y) = \sum_{j=0}^{\infty} b_{j+1}y^j$, $g^{(i)}(y)$ is a power series in y with integer coefficients, $\sum_{i=1}^{\infty} g^{(i)}(y)x^i/i!$ is a Hurwitz series in x (in the sense that the coefficient of $x_i/i!$ is a power series with integer coefficients). Thus,

$$\exp\left[\sum_{i=1}^{\infty} g^{(i)}(y)\frac{x^i}{i!}\right]$$

is also a Hurwitz series in x, which proves the theorem.

Remarks: We have that $g(x)$ is a Hurwitz series. Using the fact that $[g(x)]^k/k!$ is also a Hurwitz series for any nonnegative integer k, we define the integers $A(n, k)$ by

$$\sum_{n=k}^{\infty} A(n, k)x^n/n! = [g(x)]^k/k!. \tag{3}$$

Then, from (1), we have

$$P_n = \sum_{k=0}^{n} A(n, k), \quad P_0 = 1. \tag{4}$$

From (2), we have

$$\begin{aligned}\sum_{n=k}^{\infty} D_{n,k}y^n/n! &= (y^k/k!)\sum_{i=0}^{\infty}(-1)^i\{g(y)\}^i/i!\\ &= \sum_{i=0}^{\infty}(-1)^i\sum_{j=i}^{\infty}A(j, i)y^{j+k}/k!j!\\ &= \sum_{i=0}^{\infty}(-1)^i\sum_{n=i+k}^{\infty}A(n-k, i)\binom{n}{k}y^n/n!\\ &= \sum_{n=k}^{\infty}\sum_{i=0}^{n-k}\binom{n}{k}(-1)^i A(n-k, i)y^n/n!,\end{aligned}$$

and consequently,

$$D_{n,k} = \binom{n}{k}\sum_{i=0}^{n-k}(-1)^i A(n-k, i).$$

For tabulation purposes, we may obtain a recurrence relation for the integers $D_{n,k}$. Using (2), we have

$$D(u, y) = \sum_{n,k} D_{n,k}u^k y^n/n! = e^{-g(y)+uy}. \tag{5}$$

By differentiating both sides of (5) with respect to y, we obtain

$$\frac{\partial}{\partial y}D(u, y) = -e^{-g(y)}g'(y)e^{uy} + e^{-g(y)}e^{uy}u = D(u, y)\{-g'(y) + u\}.$$

Equating coefficients of $u^k y^n/n!$, we obtain

$$D_{n+1,k} = D_{n,k-1} - \sum_{i=0}^{n}\binom{n}{i}b_{n-i+1}(n-i+1)!D_{i,k} \quad \text{for } n, k \geqslant 0,$$

with $D_{0,0} = 1$ and $D_{n,k} = 0$ for $k > n$ or $k < 0$.

It may be noted that $f(x) = e^{g(x)}$ counts permutations in which a cycle of length j is weighted b_j.

3. SPECIAL CASES

We shall now give some special cases of $g(x)$ for which the numbers P_n are of great interest in Combinatorics.

a. $g(x) = \sum_{j \in S} x^j/j$ where S is any set of positive integers.

Then $f(x) = e^{g(x)}$ counts permutations with all cycle lengths in S. For $S = \{1, 2\}$, $g(x) = x + x^2/2$, and the numbers

$$P_n = t_n = \sum_{k=[n/2]}^{n} A(n, k)$$

have been studied by Moore [3], Moser and Wyman [7], and others. From [4], we have a congruence for t_n which is a special case of our theorem.

b. $$g(x) = \sum_{j=r}^{s}\frac{(s)_j}{(j-1)!}\frac{x^j}{j} = (1+x)^s - \sum_{j=0}^{r-1}\frac{(s)_j}{(j-1)!}\frac{x^j}{j},$$

r, s integers, $1 \leqslant r < s$.

Then $A(n, k)$ have occurred as coefficients in the k-fold convolution of binomial distributions truncated at the point $r - 1$ (see [1]). In the case in which $r = 1$, $A(n, k) = (1/n!)[\Delta^k(sx)_n]_{x=0}$ (see [2]), and the numbers

$$P_n = \sum_{k=[n/s]}^{n} A(n, k)$$

occur in combinatorial analysis being in fact P_n is equal to the sum of the horizontal entries of the table of Jordan (see [5, p. 223]).

c. $g(x) = (s-1)x + s\sum_{j=2}^{\infty} x^j/j = -x - s\log(1-x)$, s an integer, $s \geqslant 1$.

Then P_n is equal to the generalized derangement numbers $d(n, s)$ [for $s = 1$, we have the derangement number $d(n)$].

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