ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-592 *Proposed by Herta T. Freitag, Roanoke, VA*

Find all integers a and b, if any, such that $F_a L_b + F_{a-1} L_{b-1}$ is an integral multiple of 5.

B-593 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1}L_n + F_nL_{n+1}$. Prove that $A(15n - 8)$ is an integral multiple of 1220 for all positive integers n.

B-594 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$A(n) = F_{n+1}L_n + F_nL_{n+1} \quad \text{and} \quad B(n) = \sum_{j=1}^{n}\sum_{k=1}^{j} A(k).$$

Prove that $B(n) \equiv 0 \pmod{20}$ when $n \equiv 19$ or $29 \pmod{60}$.

B-595 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that

$$\sum_{k=0}^{n} k^3(n-k)^2 \equiv \binom{n+4}{6} + \binom{n+1}{6} \pmod 5.$$

<u>B-596</u> *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$S(n, k, m) = \sum_{i=1}^{m} F_{ni+k}.$$

For positive integers a, m, and k, find an expression of the form XY/Z for $S(4a, k, m)$, where X, Y, and Z are Fibonacci or Lucas numbers.

<u>B-597</u> *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Do as in Problem B-596 for $S(4a + 2, k, 2b)$ and for $S(4a + 2, k, 2b - 1)$, where a and b are positive integers.

SOLUTIONS

Fibonacci-Lucas Hyperbola for Odd n

<u>B-568</u> *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_1, L_1), (F_3, L_3), \ldots, (F_{2n+1}, L_{2n+1}), \ldots .$$

Solution by C. Georghiou, University of Patras, Greece

It is easy to show that the given points do not lie on a straight line. However,

$$L_{2n+1}/F_{2n+1} \to \sqrt{5} \text{ as } n \to \infty,$$

and it is also known that

$$5F_{2n+1}^2 - L_{2n+1}^2 = 4.$$

Therefore, the given points lie on a branch of the hyperbola with equation

$$5x^2 - y^2 = 4.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Tad P. White, and the proposer.

Fibonacci-Lucas Hyperbola for Even n

<u>B-569</u> *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_0, L_0), (F_2, L_2), \ldots, (F_{2n}, L_{2n}), \ldots .$$

Solution by J. Z. Lee, Chinese Culture University, Taipei, Taiwan, R.O.C. & J. S. Lee, National Taipei Business College, Taipei, Taiwan, R.O.C.

A simple curve passing through all of the points (F_2, L_2), (F_4, L_4), $\ldots$, (F_{2n}, L_{2n}), $\ldots$ is $y^2 - 5x^2 = 4$, since $L_n^2 - 5F_n^2 = 4(-1)^n$.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, Tad P. White, and the proposer.

Fibonacci Squareroot Triangle with Fixed Area

B-570 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a, b, and c be the positive square roots of F_{2n-1}, F_{2n+1}, and F_{2n+3}, respectively. For $n = 1, 2, \ldots$, show that

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 4.$$

Solution by L. A. G. Dresel, University of Reading, England

Let

$$P = (a + b + c)(-a + b + c)(a - b + c)(a + b - c).$$

Then, since

$$(a + b + c)(a - b + c) = (a + c)^2 - b^2,$$

and

$$(a + b - c)(-a + b + c) = b^2 - (a - c)^2,$$

we have

$$\begin{aligned} P &= (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2) \\ &= 4a^2c^2 - (a^2 + c^2 - b^2)^2 \\ &= 4F_{2n-1}F_{2n+3} - (F_{2n-1} + F_{2n+3} - F_{2n+1})^2 \\ &= 4F_{2n-1}F_{2n+3} - 4F_{2n+1}^2, \end{aligned}$$

since

$$F_{2n+3} = 3F_{2n+1} - F_{2n-1}.$$

Using the Binet forms, we find that $F_{2n-1}F_{2n+3} = F_{2n+1}^2 + 1$; hence, $P = 4$.

We note in passing that Heron's formula gives the area of a triangle of sides a, b, c as $\frac{1}{4}\sqrt{P}$, and therefore the area of a triangle whose sides are the positive square roots of F_{2n-1}, F_{2n+1}, and F_{2n+3} will be $\frac{1}{2}$ for $n = 1, 2, 3, \ldots$.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Weighted Rising Diagonal Sum

B-571 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Conjecture and prove a simple expression for

$$\sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r}$$

where $[n/2]$ is the largest integer m with $2m \leqslant n$.

Solution by Philip L. Mana, Albuquerque, NM

Let S be the given sum and $q = [n/2]$. Then

$$S = \sum_{r=0}^{q}\binom{n-r}{r} + \sum_{r=0}^{q}\frac{r}{n-r}\binom{n-r}{r} = F_{n+1} + \sum_{n=1}^{q}\binom{n-r-1}{r-1}$$

$$= F_{n+1} + F_{n-1} = L_n,$$

using the rising diagonal formula

$$\sum_{r=0}^{q}\binom{n-r}{r} = F_{n+1}.$$

Also solved by Paul S. Bruckman, Oroardo Brugia & Piero Filipponi, L. A. G. Dresel, Herta T. Freitag, C. Goerghiou, L. Kuipers, J. Z. Lee & J. S. Lee, F. S. Makri & D. Antzoulakos, Bob Prielipp, J. Suck, Tad P. White, and the proposer.

Continued Fraction

B-572 *Proposed by Ambati Jaya Krishna, Student, Johns Hopkins University, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC*

Evaluate the continued fraction:

$$1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cdots}}}$$

Solution by C. Georghiou, University of Patras, Greece

This is the same as Problem H-394 in this *Quarterly* (Vol. 24, no. 1 [1986], p. 88] proposed by the same authors. Its solution is as follows:

From the theory of continued fractions, it is known that (See, for example, M. Abramowitz & A. Stegun, *Handbook of Mathematical Functions* [New York: Dover, 1970], p. 19):

$$g_n(x) = \frac{1}{a_0} - \frac{x}{a_0a_1} + \frac{x^2}{a_0a_1a_2} - \cdots + (-1)^n\frac{x^n}{a_0a_1a_2\cdots a_n}$$

$$= \frac{1}{a_0} + \frac{a_0x}{a_1 - x} + \frac{a_1x}{a_2 - x} + \cdots + \frac{a_{n-1}x}{a_n - x}.$$

Take $a_n = 2n + 2$ and $x = 1$. Then, the n^{th} convergent of the given continued fraction, f_n, is given by

$$f_n = \frac{1}{g_n} - 2.$$

Since $\lim_{n\to\infty} g_n = 1 - e^{-1/2}$, we get $f = (e^{1/2} - 1)^{-1}$.

Also solved by Paul S. Bruckman, L. Kuipers & Peter S. J. Shiue, J. Z. Lee & J. S. Lee, and the proposer.

B-573 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

For all nonnegative integers n, prove that

$$\sum_{k=0}^{n}\binom{n}{k}L_kL_{n-k} = 4 + 5\sum_{k=0}^{n}\binom{n}{k}F_kF_{n-k}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that

$$S = \sum_{k=0}^{n}\binom{n}{k}(L_kL_{n-k} - 5F_kF_{n-k}) = 4,$$

which is equivalent to the required result.

$$\begin{aligned} L_kL_{n-k} - 5F_kF_{n-k} &= (\alpha^k + \beta^k)(\alpha^{n-k} + \beta^{n-k}) - (\alpha^k - \beta^k)(\alpha^{n-k} - \beta^{n-k}) \\ &= 2\alpha^k\beta^{n-k} + 2\beta^k\alpha^{n-k}. \end{aligned}$$

Thus,

$$\begin{aligned} S = 2\sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k} + 2\sum_{k=0}^{n}\binom{n}{k}\beta^k\alpha^{n-k} &= 4(\alpha + \beta)^n \quad \text{[by the Binomial Theorem]} \\ &= 4 \cdot 1^n = 4. \end{aligned}$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Sahib Singh, J. Suck, Tad P. White, and the proposer.

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