ON SOME MIXTURES OF DISTRIBUTIONS OF ORDER k

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1. INTRODUCTION

Problems associated with the frequency of occurrence of runs of like elements in a series of Bernoulli trials have recently attracted quite a lot of attention. The reasons may possibly be traced not only to the theoretical interest they present as generalizing the usual binomial set up, but also to the practical value that any theoretical results in this direction would have with regard to statistical hypothesis testing. Feller [3] considered a series of Bernoulli trials and concentrated on the relationship between the probability distributions of the number of runs of k successes in n trials ($N_{k,n}$) and the number of trials needed to get r runs of k successes ($T_{k,r}$). He showed that

$$P(N_{k,n} \geqslant r) = P(T_{k,r} \leqslant n),\ r = 0,\ 1,\ \ldots,\ \left[\frac{n}{k}\right]$$

and examined the asymptotic behavior of the distributions of $N_{k,n}$ and $T_{k,r}$. Fréchet [4] led the way in considering the problem of deriving the exact distribution of $N_{k,n}$ and $T_{k,1}$ using his theory on the probability of the conjunction of events. More recently, Shane [21] and Turner [23] obtained expressions for the probability distribution of $T_{k,1}$ using the polynacci polynomials of order k and the entries of the Pascal triangle, respectively. Philippou & Muwafi [19] provided an alternative formula for this probability distribution in terms of the multinomial coefficients. Also, Uppuluri & Patil [24] gave an explicit expression in terms of weighted binomial coefficients that was implicit in the work of Philippou *et al.* [16]. Philippou *et al.* [17] obtained the exact distribution of $T_{k,r}$ ($r \geqslant 1$) by pointing out that $T_{k,r}$ can be represented by the sum of r independent and identically distributed random variables whose distribution coincides with that of $T_{k,1}$ (see also Philippou [15]). The exact distributions of $T_{k,r}$ and $T_{k,1}$ are called the "negative binomial distribution of order k" and "geometric distribution of order k," respectively. Hirano [6] and Philippou & Makri [18] employed the combinatorial argument of Philippou & Muwafi [19] to derive the exact distribution of $N_{k,n}$ which they named "the binomial distribution of order k." Certain limiting cases and/or mixtures of the above distributions have also been examined. Philippou *et al.* [17] showed that the distribution of $T_{k,r} - kr$ as $r \to +\infty$ reduces to a certain form of generalized Poisson distribution examined in further detail by Philippou [14], who

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names it "the Poisson distribution of order k" (as being the limit of an "order k" distribution). In addition, Philippou (in [14]) discussed a gamma compound (mixed) Poisson distribution of order k. Aki *et al.* [1] derived a logarithmic distribution of order k as the limiting distribution of the random variable

$$T_{k,r} \mid (T_{k,r} > kr) \text{ as } r \to 0$$

(see also the work of Hirano *et al.* [7] who gave figures of distributions of order k). Finally, Panaretos & Xekalaki [13] defined and studied some other distributions of order k. These are the hypergeometric and the negative hypergeometric distributions of order k, a limiting case of the zero-truncated compound Poisson distribution of order k (the logarithmic series distribution of order k) as well as the Polya, the inverse Polya, and the generalized Waring distributions of order k.

As is well known, the number of applications of the above-mentioned distributions when $k = 1$ (ordinary binomial, geometric, or negative binomial distributions) is vast. However, applying these distributions presupposes a constant probability of success p which is a requirement that can hardly hold in practice. So, in many instances, combinations of different binomial, geometric, or negative binomial distributions have been considered. That is, p is allowed to vary from trial to trial according to some probability law thus giving rise to compound (mixed) forms of these distributions. The particular case of a beta distributed p gives rise to distributions belonging to the class of inverse factorial series distributions that have played an important role in the medical and biological fields. Two such distributions are the beta-compound geometric, also known as the Yule distribution (see [32]), and the beta-compound negative binomial distribution, also known as the generalized Waring distribution (see Xekalaki [25]). Their applications, however, are not confined to these fields. They have also been applied to fields such as accident, income, or geographical analysis, linguistics, bibliographic research, and reliability. A selection of their contribution to these fields can be found in Dacey [2], Haight [5], Irwin [8, 9, 10], Kendall [11], Krishnaji [12], Schubert & Glänzel [20], Simon [22], Xekalaki [26-30], and Xekalaki & Panaretos [31].

In this paper we consider generalizations of beta-geometric and beta-negative binomial distribution. These are obtained in Sections 2 and 3 as mixtures of the Poisson distribution of order k, in a manner similar to the derivation of the geometric and the negative binomial distributions as mixtures of the ordinary Poisson distribution. Expressions for their probabilities and the first two moments are given. In Section 4 it is shown that the Poisson and the gamma-compound Poisson distributions of order k are limiting cases of the generalized beta-negative binomial so that the theory of those distributions that are of negative binomial form is a particular case of that shown in Section 4.

Before providing the main results, let us introduce some notation and terminology.

A nonnegative, integer-valued random variable (r.v.) X is said to have the beta-geometric (Yule) distribution with parameter c if its probability function (p.f.) is given by

$$P(X = x) = \frac{cx!}{(c+1)_{(x+1)}}, \quad c > 0, \ x = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$a_{(\beta)} = \Gamma(a + \beta)/\Gamma(a), \ a > 0, \ \beta \in R.$$

A nonnegative, integer-valued r.v. X is said to have the beta-negative binomial (generalized Waring) distribution with parameters a, b, c if its p.f. is

$$P(X = x) = \frac{c_{(b)}}{(a + c)_{(b)}} \frac{a_{(x)}b_{(x)}}{(a + b + c)_{(x)}} \frac{1}{x!} \tag{1.2}$$

$$a, b, c > 0, x = 0, 1, 2, \ldots .$$

Their probability generating functions (p.g.f.) are of the form

$${}_2F_1(\alpha, \beta; \gamma; s)/{}_2F_1(\alpha, \beta; \gamma; 1),$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by the series

$${}_2F_1(\alpha, \beta; \gamma; s) = \sum_{r=0}^{\infty} \frac{\alpha_{(r)}\beta_{(r)}}{\gamma_{(r)}} \frac{s^r}{r!}, \tag{1.3}$$

which is convergent for all $|s| \leqslant 1$ provided that $\gamma > \alpha + \beta$.

In the sequel, we will refer to the distribution with p.g.f.

$$H_{(s)} = p^r(1 - qs)^{-r}, \; r \geqslant 1, \tag{1.4}$$

as the negative binomial distribution. For $r = 1$ the resulting distribution will be termed "the geometric distribution."

A continuous r.v. X will be said to have the beta distribution of the first kind with parameters a, b [beta I (a, b)] if its probability density function (p.d.f.) is given by

$$f(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}, \; a > 0, \; b > 0, \; 0 < x < 1. \tag{1.5}$$

Finally, a continuous r.v. X will be said to have the beta distribution of the second kind with parameters a, b [beta II (a, b)] if its p.d.f. is given by

$$h(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 + x)^{-(a+b)}, \; a, b > 0, \; x > 0. \tag{1.6}$$

2. THE BETA-GEOMETRIC DISTRIBUTION OF ORDER k

As implied by its name, the beta-geometric (Yule) distribution defined by (1.1) is obtained as a mixture on p of the geometric distribution when p is a beta r.v. In fact, that was the theoretical model on which Yule [32] derived this distribution. In particular, if $\hat{\theta}$ denotes the mixing with respect to a parameter θ and ~ denotes equivalence, then

$$\text{beta-geometric } (c) \sim \text{geometric } (p) \underset{\hat{p}}{} \text{ beta I } (c, 1).$$

Since the geometric distribution arises as an exponential mixture of the Poisson distribution, this model is equivalent to

$$\text{beta-geometric } (c) \sim \text{Poisson } (\lambda) \underset{\hat{\lambda}}{} \text{ exponential } (1/b) \underset{\hat{b}}{} \text{ beta II } (1, c).$$

The structure of the latter model reveals the possibility of extending the beta-geometric distribution by replacing the Poisson distribution by a generalized Poisson distribution.

Consider a r.v. X which, conditional on some other r.v. λ ($\lambda > 0$), has a generalized Poisson distribution. Then its p.g.f. is of the form

$$G_{X|\lambda}(s) = \exp\{\lambda(g(s) - 1)\}$$

where $g(s)$ is a valid p.g.f., or, equivalently (see Feller [3], p. 291) of the form

$$G_{X|\lambda}(s) = \exp\left\{\sum_{i=1}^{k} \lambda_i(s^i - 1)\right\}, \tag{2.1}$$

$$\lambda_i = \lambda g^{(i)}(0)/i!, \; k \in \mathrm{I}^+ \cup \{+\infty\}, \; \sum_{i=1}^{k} \lambda_i < +\infty.$$

Assume that $k < +\infty$ and that $\lambda_i = \lambda_j$, $i \neq j$, $i, j = 1, 2, \ldots, k$, i.e., that $g(s)$ is the p.g.f. of the discrete uniform distribution on $\{1, 2, \ldots, k\}$. Then $G_{X|\lambda}(s)$ is the p.g.f. of the Poisson distribution with parameter λ/k generalized by the uniform distribution on $\{1, 2, \ldots, k\}$, i.e.,

$$G_{X|\lambda}(x) = \exp\left\{\frac{\lambda}{k}\sum_{i=1}^{k}(s^i - 1)\right\}. \tag{2.2}$$

The probability distribution defined by (2.2) is known in the literature as the Poisson distribution of order k (Philippou *et al.* [17]). Thus, we have shown that the Poisson distribution of order k with parameter λ can be viewed as the distribution of $X_1 + X_2 + \cdots + X_N$, where N is a Poisson (λk) r.v. and X_1, X_2, ... are independent r.v.'s that are distributed on $\{1, 2, \ldots, k\}$ uniformly and independently of N.

Suppose now that λ has an exponential distribution whose parameter is itself a r.v. having a beta II $(1, c)$ distribution, i.e., the p.d.f. of λ is of the form

$$f(\lambda) = \int_0^{+\infty} \frac{c}{m} e^{-(1/m)\lambda}(1 + m)^{-(c+1)} dm.$$

Then the unconditional distribution of X has p.g.f.

$$\begin{aligned} G_X(s) &= c\int_0^{+\infty}\int_0^{+\infty} m^{-1}(1 + m)^{-(c+1)}\exp\left\{-\lambda\left(\frac{1}{m} + k - \sum_{i=1}^{k} s^i\right)\right\} dm\, d\lambda \\ &= c\int_0^{+\infty}(1 + m)^{-(c+1)}\left(1 + m\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-1} dm, \end{aligned}$$

i.e.,

$$G_X(s) = \frac{c}{c+1}\,{}_2F_1\left(1, 1; c + 2; \sum_{i=1}^{k} s^i - k + 1\right). \tag{2.3}$$

For $k = 1$, (2.3) reduces to

$$G_X(s) = \frac{c}{c+1}\,{}_2F_1(1, 1; c + 2; s)$$

which is the p.g.f. of the beta-geometric distribution. Hence, (2.3) is a generalized form of the beta-geometric distribution. In the sequel, we will refer to this distribution as the beta-geometric distribution of order k with parameter c.

The first two factorial moments of the beta-geometric distribution of order k can be obtained using (2.3); thus,

$$\begin{aligned} E(X) &= \frac{c}{c+1}\frac{1}{c+2}\,{}_2F_1(2, 2; c + 3; 1)\sum_{i=1}^{k} i \\ &= \frac{c}{(c+1)}\frac{1}{(c+2)}\frac{(c+2)_{(2)}}{(c-1)_{(2)}}\frac{k(k+1)}{2} = \frac{k(k+1)}{2(c-1)}; \end{aligned} \tag{2.4}$$

$$\begin{aligned} E(X(X - 1)) &= \frac{c}{c+1}\frac{4}{(c+2)_{(2)}}\,{}_2F_1(3, 3; c + 4; 1)\left(\sum_{i=1}^{k} i\right)^2 \\ &\quad + \frac{c}{c+1}\frac{1}{(c+2)}\,{}_2F_1(2, 2; c + 3; 1)\sum_{i=2}^{k} i(i - 1) \\ &= \frac{k^2(k+1)^2}{(c-1)(c-2)} + \frac{k(k+1)(k-1)}{3(c-1)}. \end{aligned}$$

Hence, the variance is

$$V(X) = \frac{k^2(k+1)^2c^2}{4(c-1)^2(c-2)} + \frac{k(k^2-1)(3k+2)}{36(c-1)}. \tag{2.5}$$

Note that both the mean and the variance of the beta-geometric distribution are greater than or equal to the corresponding mean and variance of the ordinary beta-geometric distribution and do not exist when $c \leqslant 1$ and $c \leqslant 2$, respectively.

Because of their simplicity, relationships (2.4) and (2.5) can be of great practical value as far as moment estimation of the parameter c is concerned, especially because of the complexity of the maximum likelihood method for generalized hypergeometric-type distributions. Thus, based on a random sample of size n, the moment estimator of c is

$$\hat{c} = \frac{k(k+1)}{2\overline{X}} + 1 \tag{2.6}$$

with variance

$$V(\hat{c}) = \frac{c^2(c-1)^2}{n(c-2)} + \frac{(c-1)(k-1)(3k+2)}{9nk(k+1)}, \tag{2.7}$$

where $\overline{X}$ is the sample mean.

Now, we shall show that if X is a r.v. having the beta-geometric distribution of order k with parameter $c > 0$, its p.f. is given by

$$P(X = x) = c\sum_{\ell=0}^{\infty} \frac{(1-k)^{\ell}}{\ell!} \sum_{\Sigma i x_i = x} \frac{((\Sigma x_j + \ell)!)^2}{(c+1)_{(\Sigma x_j + \ell + 1)}} \frac{1}{\prod_{j=1}^{k} x_j!}, \tag{2.8}$$

$$x = 0, 1, 2, \ldots .$$

From (2.3), we have that

$$G_X(s) = \frac{c}{c+1}\, {}_2F_1\Big(1, 1; c+2; \sum_{i=1}^{k} s^i + 1 - k\Big),$$

i.e.,

$$\begin{aligned} G_X(s) &= \frac{c}{c+1} \sum_{r=0}^{\infty} \frac{r!}{(c+2)_{(r)}} \left(\sum_{i=1}^{k} s^i + 1 - k\right)^r \\ &= \frac{c}{c+1} \sum_{r=0}^{\infty} \frac{(r!)^2}{(c+2)_{(r)}} \sum_{\ell + \Sigma r_i = r} \binom{r}{r_1, r_2, \ldots, r_k, \ell} \frac{(1-k)^{\ell} \prod_{i=1}^{k} s^{j r_j}}{r!} \\ &= \frac{c}{c+1} \sum_{r=0}^{\infty} \sum_{\ell=0}^{r} \sum_{\Sigma r_i = r - \ell} \frac{((\Sigma r_i + \ell)!)^2 (1-k)^{\ell} s^{\Sigma i r_i}}{(c+2)_{(\Sigma r_i + \ell)}\, \ell! \prod_{j=1}^{k} r_j!}. \end{aligned}$$

Setting

$$r_i = x_i, \quad i = 1, 2, \ldots, k, \quad \text{and} \quad r + \sum_{i=1}^{k} (i-1) r_i = x,$$

we obtain

$$G_X(s) = \frac{c}{c+1} \sum_{x=0}^{\infty} s^x \sum_{\ell=0}^{\infty} \frac{(1-k)^{\ell}}{\ell!} \sum_{\Sigma i x_i = x} \frac{\left(\left(\sum_{j=1}^{k} x_j + \ell\right)!\right)^2}{(\rho + 2)_{(\Sigma x_j + \ell)} \prod_{j=1}^{k} x_j!}$$

from which (2.8) follows.

3. THE BETA-NEGATIVE BINOMIAL DISTRIBUTION OF ORDER k

The beta-negative binomial distribution (or generalized Waring distribution) was considered by Irwin [9] in the context of problems in accident analysis. It was obtained from the theoretical model

beta-negative binomial $(a, b; c) \sim$ negative binomial $(b, p) \underset{p}{\wedge}$ beta I (c, a)

which is equivalent to

beta-negative binomial $(a, b; c) \sim$ Poisson $(\lambda) \underset{\lambda}{\wedge}$ gamma $\left(\frac{1}{m}, b\right) \underset{m}{\wedge}$ beta II (a, c)

Then, an extension of the beta-negative binomial distribution can be defined by a slight modification of the latter mechanism.

Let X be a r.v. such that, conditional on another nonnegative r.v., λ has a Poisson distribution of order k with parameter λ and p.g.f. given by (2.3). Assume now that λ has a gamma distribution whose scale parameter is a beta II (a, c) r.v., i.e., assume that λ has a p.d.f. of the form

$$f(\lambda) = \frac{\Gamma(a + c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \lambda^{b-1} \int_0^{+\infty} m^{a-b-1}(1 + m)^{-(a+c)} e^{-\lambda/m}\, dm.$$

Then the final resulting distribution of X will have p.g.f.

$$G_X(s) = \frac{\Gamma(a + c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^{+\infty}\int_0^{+\infty} m^{a-b-1}(1 + m)^{-(a+c)}\lambda^{b-1} \times \exp\left\{-\lambda\left(\frac{1}{m} + k - \sum_{i=1}^{k} s^i\right)\right\} d\lambda\, dm$$

$$= \frac{\Gamma(a + c)}{\Gamma(a)\Gamma(c)} \int_0^{+\infty} m^{a-1}(1 + m)^{-(a+c)}\left(1 + m\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b} dm,$$

i.e.,

$$G_X(s) = \frac{c_{(b)}}{(a + c)_{(b)}}\, {}_2F_1\left(a, b; a + b + c; \sum_{i=1}^{k} s^i - k + 1\right). \tag{3.1}$$

The above relationship reduces, for $k = 1$, to

$$G_X(s) = \frac{c_{(b)}}{(a + c)_{(b)}}\, {}_2F_1(a, b; a + b + \rho; s),$$

i.e., it coincides with the p.g.f. of the usual beta-negative binomial distribution. Thus (3.1) defines a more general form of beta-negative binomial distribution in the framework of distributions of order k. We will refer to this distribution as the beta-negative binomial distribution of order k with parameters a, b, and c; $a, b, c > 0$.

The mean of this distribution can be obtained from (3.1) by differentiation at $s = 1$.

$$E(X) = \frac{c_{(b)}}{(a + c)_{(b)}} \frac{ab}{a + b + c}\, {}_2F_1(a + 1, b + 1; a + b + c + 1; 1) \sum_{i=1}^{k} i$$

$$= \frac{c_{(b)}}{(a + c)_{(b)}} \frac{ab}{a + b + c} \frac{(a + c)_{(b+1)}}{(c - 1)_{(b)}} \frac{k(k + 1)}{2}$$

i.e.,

$$E(X) = \frac{abk(k + 1)}{2(c - 1)}. \tag{3.2}$$

The second factorial moment is

$$\begin{aligned}\mu_{[2]} &\equiv E(X(X-1)) \\ &= \frac{c_{(b)}}{(a+c)_{(b)}}\frac{a_{(2)}b_{(2)}}{(a+b+c)_{(2)}}\,{}_2F_1(a+2,\ b+2;\ a+b+c+2;\ 1)\left(\sum_{i=1}^{k} i\right)^2 \\ &\quad + \frac{c_{(b)}}{(a+c)_{(b)}}\frac{ab}{a+b+c}\,{}_2F_1(a+1,\ b+1;\ a+b+c+1;\ 1)\sum_{i=1}^{k} i(i-1),\end{aligned}$$

i.e.,

$$\mu_{[2]} = \frac{ab(a+1)(b+1)k^2(k+1)^2}{4(c-1)(c-2)} + \frac{abk(k^2-1)}{3(c-1)}. \tag{3.3}$$

Hence, we have, for the variance,

$$V(X) = \frac{k^2(k+1)^2ab(c+a-1)(c+b-1)}{4(c-1)^2(c-2)} + \frac{abk(k^2-1)(3k+2)}{36(c-1)}. \tag{3.4}$$

Because application of the distribution will require estimation of three parameters (a, b, and c), we also provide the third factorial moment.

$$\begin{aligned}\mu_{[3]} \equiv E(X(X-1)(X-2)) &= \frac{a(a+1)(a+2)b(b+1)(b+2)k^3(k+1)^3}{8(c-1)(c-2)(c-3)} \\ &\quad + \frac{a(a+1)b(b+1)k^2(k^2-1)(k+1)}{2(c-1)(c-2)} \\ &\quad + \frac{abk(k^2-1)(k-2)}{4(c-1)}.\end{aligned} \tag{3.5}$$

Equations (3.2), (3,3), and (3.5) can be used to develop estimators of the parameters a, b, and c if a moment method of estimation is to be considered.

Note that for $k = 1$ we obtain from equations (3.2)-(3.5) the corresponding moments of the usual beta-negative binomial distribution. Inspection of these formulas shows that $\mu_{[i]}$ is expressed in terms of the first i factorial moments of the beta-negative binomial distribution, $i = 1, 2, 3$. Hence $\mu_{[i]}$ exists only if $c > i$, $i = 1, 2, 3$.

Let us now consider a nonnegative, integer-valued r.v. X whose probability distribution is the beta-negative binomial distribution of order k. We will show that the p.f. of X is given by

$$P(X = x) = \frac{c_{(b)}}{(a+c)_{(b)}}\sum_{\ell=0}^{+\infty}\frac{(1-k)^{\ell}}{\ell!}\sum_{\Sigma i x_i = x}\frac{a_{(\Sigma x_i+\ell)}b_{(\Sigma x_i+\ell)}}{(a+b+c)_{(\Sigma x_i+\ell)}\prod_{j=1}^{k} x_j!}, \tag{3.6}$$

$$x = 0,\ 1,\ 2,\ \ldots\ .$$

Setting $c^* = \dfrac{c_{(b)}}{(a+c)_{(b)}}$ we have, from (3.1),

$$\begin{aligned}G_X(s) &= c^*\,{}_2F_1\left(a,\ b;\ a+b+c;\ \sum_{i=1}^{k} s^i + 1 - k\right) \\ &= c^*\sum_{r=0}^{\infty}\frac{a_{(r)}b_{(r)}}{(a+b+c)_{(r)}}\ \frac{\left(\sum_{i=1}^{k} s^i + 1 - k\right)^r}{r!}\end{aligned}$$

$$= c^* \sum_{r=0}^{\infty} \frac{a_{(r)} b_{(r)}}{(a+b+c)_{(r)}} \sum_{\ell + \Sigma r_i = r} \binom{r}{\ell, r_1, \ldots, r_k} \frac{(1-k)^{\ell} \prod_{j=1}^{k} s^{j r_j}}{r!}$$

$$= c^* \sum_{r=0}^{\infty} \sum_{\ell=0}^{r} \sum_{\Sigma r_i = r - \ell} \frac{a_{(\Sigma r_i + \ell)} b_{(\Sigma r_i + \ell)} (1-k)^{\ell} s^{\Sigma i r_i}}{(a+b+c)_{(\Sigma r_i + \ell)} \ell! \prod_{j=1}^{k} r_j!}.$$

Let $x_i = r_i$, $i = 1, 2, \ldots, k$, and $x = r + \sum_{i=1}^{k} (i-1) r_i$. Then the p.g.f. of X becomes

$$G_X(x) = c^* \sum_{x=0}^{+\infty} s^x \sum_{\ell=0}^{\infty} \frac{(1-k)^{\ell}}{\ell!} \sum_{\Sigma i x_i = x} \frac{a_{(\Sigma x_i + \ell)} b_{(\Sigma x_i + \ell)}}{(a+b+c)_{(\Sigma x_i + \ell)} \prod_{j=1}^{k} x_j!}$$

which leads to (3.6).

It is interesting to observe that the beta-geometric distribution of order k defined in Section 2 and the beta-negative binomial distribution of order k defined in Section 3 are related in the same manner in which the ordinary beta-geometric and beta-negative binomial distributions are related. In particular, the beta-geometric distribution of order k can be thought of as a special case of the beta-negative binomial distribution of order k for $a = b = 1$.

4. SOME LIMITING CASES OF THE BETA-NEGATIVE BINOMIAL DISTRIBUTION OF ORDER k

It is known (see Irwin [8]) that the beta-negative binomial distribution can take a negative binomial or a Poisson form for certain limiting values of its parameters. So, naturally one would inquire whether its generalization as defined in Section 3, i.e., the beta-negative binomial distribution of order k tends to a negative binomial or Poisson type of distribution of the same order. It can be shown that, indeed, this is the case.

The Poisson distribution of order k and the gamma-compound Poisson distribution of order k are obtained as limiting cases of the beta-negative binomial distribution of order k as indicated by the following theorems.

Theorem 4.1: Let X be a nonnegative, integer-valued r.v. whose probability distribution is the beta-negative binomial of order k with parameters a, b, c. Then

$$\lim_{H} G_X(s) = \exp\left\{\frac{ab}{a+c}\left(\sum_{i=1}^{k} s^i - k\right)\right\}, \tag{4.1}$$

where $\lim_{H}$ stands for limit as $a \to +\infty$, $b \to +\infty$, $c \to +\infty$ so that $ab/(a+c) < +\infty$ and $a/(a+c) \to 0$.

The result of this theorem was not unexpected since, by its derivation, the beta-negative binomial distribution of order k can be regarded as a beta mixture of the gamma-compound Poisson distribution of order k (studied by Philippou [14]) with p.g.f.

$$G(s) = \left(1 + m\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b}, \quad m > 0,\ b > 0, \tag{4.2}$$

which converges to a Poisson distribution of order k as demonstrated by the following theorem.

Theorem 4.2: Let X be a r.v. having the gamma-compound Poisson distribution of order k with p.g.f. $G(s)$ given by (4.2). Then,

$$G(s) \to \exp\left\{mb\sum_{i=1}^{k}(s^i - 1)\right\} \tag{4.3}$$

as $m \to 0$, $b \to +\infty$ so that $mb < +\infty$.

Theorem 4.3: Let X be defined as in Theorem 4.1. Then,

$$\lim_{H'} G_X(s) = \left(1 + \frac{a}{c}\left(k - \sum_{i=1}^{k} s^i\right)\right)^{-b}, \tag{4.4}$$

where $\lim_{H'}$ stands for limit as $a \to +\infty$ and $c \to +\infty$ so that $a/(a + c) < +\infty$.

Note that, for $k = 1$, relationships (4.1) and (4.4) yield the Poisson and negative binomial limit of the ordinary beta-geometric distribution, respectively, while (4.3) yields the Poisson limit of the ordinary negative binomial distribution.

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