# ELEMENTARY PROBLEMS AND SOLUTIONS 

## Edited by

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## DEFINITIONS

The Fibonacci numbers $F_{n}$ and the Lucas numbers $L_{n}$ satisfy
and

$$
F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1
$$

$L_{n+2}=L_{n+1}+L_{n}, L_{0}=2, L_{1}=1$.
PROBLEMS PROPOSED IN THIS ISSUE
B-598 Proposed by Herta T. Freitag, Roanoke VA
For which positive integers $n$ is ( $2 L_{n}, L_{2 n}-3, L_{2 n}-1$ ) a Pythagorean triple? For which of these $n$ 's is the triple primitive?

B-599 Proposed by Herta T. Freitag, Roanoke, VA
Do B-598 with the triple now ( $2 L_{n}, L_{2 n}+1, L_{2 n}+3$ ).
B-600 Proposed by Philip L. Mana, Albuquerque, NM
Let $n$ be any positive integer and $m=n^{13}-n$. Prove that $F_{n}$ is an integral multiple of 30290 .

B-601 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy
Let $A_{n, k}=\left(F_{n}+F_{n+1}+\cdots+F_{n+k-1}\right) / k$. Find the smallest $k$ in $\{2,3,4$, $\ldots\}$ such that $A_{n, k}$ is an integer for every $n$ in $\{0,1,2, \ldots\}$.

B-602 Proposed by Paul S. Bruckman, Fair Oaks, CA
Let $H_{n}$ represent either $F_{n}$ or $L_{n}$.
(a) Find a simplified expression for $\frac{1}{H_{n}}-\frac{1}{H_{n+1}}-\frac{1}{H_{n+2}}$.
(b) Use the result of (a) to prove that

$$
\sum_{n=1}^{\infty} \frac{1}{F_{n}}=3+2 \sum_{n=1}^{\infty} \frac{1}{F_{2 n-1} F_{2 n+1} F_{2 n+2}} .
$$

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B-603 Proposed by Paul S. Bruckman, Fair Oaks, CA
Do the Lucas analogue of $B-602(\mathrm{~b})$.

## SOLUTIONS

## Downrounded Square Roots

B-574 Proposed by Valentina Bakinova, Rondout Valley, NY
Let $\alpha_{1}, a_{2}, \ldots$ be defined by $\alpha_{1}=1$ and $\alpha_{n+1}=\left[\sqrt{s_{n}}\right]$, where $s_{n}=\alpha_{1}+a_{2}+$ $\cdots+a_{n}$ and $[x]$ is the integer with $x-1<[x] \leqslant x$. Find $a_{100}, s_{100}, a_{1000}$, and $s_{1000^{\circ}}$

Solution by L.A. G. Dressel, University of Reading, England

Starting with $s_{1}=1$, we have $a_{2}=a_{3}=a_{4}=1$ and $s_{4}=4$. Suppose now that, for some integer $h, h \geqslant 2$, we have $s_{t}=h^{2}$. Then, since $(h+1)^{2}=h^{2}+2 h+1$, we obtain
$\begin{aligned} & a_{t+1}=a_{t+2}=a_{t+3}=h \quad \text { and } \quad s_{t+3}=(h+1)^{2}+h-1 ; \\ & \text { further, } \\ & a_{t+4}=a_{t+5}=h+1 \quad \text { and } \quad s_{t+5}=(h+2)^{2}+h-2,\end{aligned}$
and continuing as long as $j \leqslant h, s_{t+2 j+1}=(h+j)^{2}+h-j$, so that for $j=k$ we obtain $s_{t+2 h+1}=(2 h)^{2}$.

Since $s_{4}=2^{2}$, it follows that whenever $s_{n}$ is a perfect square it is of the form $2^{2 i}(i=0,1,2, \ldots)$, and that if

$$
s_{t_{i}}=2^{2 i} \quad \text { and } \quad s_{t_{i+1}}=2^{2(i+1)}
$$

then $t_{i+1}=t_{i}+2^{i+1}+1$.
Since $s_{1}=1, t_{0}=1$, and we can show that

$$
t_{i}=2^{i+1}+i-1, \text { for } i=0,1,2, \ldots
$$

To find $\alpha_{100}$ and $s_{100}$ : we have $t_{5}=64+4=68$, so that $s_{68}=(32)^{2}$,

$$
s_{99}=(32+15)^{2}+32-15, a_{100}=47, s_{100}=(47)^{2}+64=2273
$$

To find $a_{1000}$ and $s_{1000}: t_{8}=2^{9}+7=519$ and $s_{519}=(256)^{2}$,

$$
s_{998}=(256+239)^{2}+256-239, \alpha_{999}=\alpha_{1000}=495
$$

and

$$
s_{1000}=(256+240)^{2}+256-240=(496)^{2}+16=246032
$$

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, J. Suck, M. Wachtel, and the proposer.

## Summing Products

B-575 Proposed by L.A. G. Dresel, Reading, England
Let $R_{n}$ and $S_{n}$ be sequences defined by given values $R_{0}, R_{1}, S_{0}, S_{1}$ and the recurrence relations $R_{n+1}=r R_{n}+t R_{n-1}$ and $S_{n+1}=s S_{n}+t S_{n-1}$, where $r$, $s$, $t$ are constants and $n=1,2,3, \ldots$. Show that

Solution by J. Suck, Essen, Germany
This identity may be hard to dream up but is easy to prove by induction:
For $n=1$, the left-hand side is $(r+s) R_{1} S_{1}$, and the right-hand side is

$$
\left(r R_{1}+t R_{0}\right) S_{1}+R_{1}\left(s S_{1}+t S_{0}\right)-t\left(R_{1} S_{0}+R_{0} S_{1}\right),
$$

i.e., both are the same.

For the step from $n$ to $n+1$, we have to show that
$t\left(R_{n+1} S_{n}+R_{n} S_{n+1}\right)+(r+s) R_{n+1} S_{n+1}$
$=\left(r R_{n+1}+t R_{n}\right) S_{n+1}+R_{n+1}\left(s S_{n+1}+t S_{n}\right)$,
which, after a little sorting, is seen to be true.
Also solved by Paul S. Bruckman, L. Cseh, Piero Filipponi \& Adina Di Porto, L. Kuipers, Andreas N. Philippou \& Demetris Antzoulakos, George Philippou, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

## Product of Three Fibonacci Numbers

B-576 Proposed by Herta T. Freitag, Roanoke, VA
Let $A=L_{2 m+3(4 n+1)}+(-1)^{m}$. Show that $A$ is a product of three Fibonacci numbers for all positive integers $m$ and $n$.

Solution by Lawrence Somer, Washington, D.C.
We prove the more general result that, if $r \geqslant 1$, then

$$
L_{2 r+1}+(-1)^{r+1}=5 F_{r} F_{r+1}=F_{5} F_{r} F_{r+1} .
$$

Note that, if $2 r+1=2 m+3(4 n+1)$, then

$$
m \equiv r+1(\bmod 2) \quad \text { and } \quad(-1)^{m}=(-1)^{r+1}
$$

By the Binet formulas and using the fact that $\alpha \beta=-1$, $5 F_{r} F_{r+1}=5\left[\left(\alpha^{r}-\beta^{r}\right) / \sqrt{5}\right]\left[\left(\alpha^{r+1}-\beta^{r+1}\right) / \sqrt{5}\right]$
$=\alpha^{2 r+1}+\beta^{2 r+1}-(\alpha \beta)^{r}(\alpha+\beta)$

$$
=L_{2 r+1}-(-1)^{r} L_{1}=L_{2 r+1}+(-1)^{r+1},
$$

and we are done.

Also solved by Paul S. Bruckman, L.A. G. Dresel, Piero Filipponi, George Koutsoukellis, L. Kuipers, Andreas N. Philippou \& Demetris Antzoulakos, Bob Prielipp, H.-J. Seiffert, Sahib Singh, J. Suck, and the proposer.

## Difference of Squares

B-577 Proposed by Herta T. Freitag, Roanoke, VA
Let $A$ be as in B-575. Show that $4 A / 5$ is a difference of squares of Fibonacci numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI
Let $m$ and $n$ be arbitrary positive integers. We shall show that

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$$
\begin{equation*}
4 A / 5=F_{m+6 n+3}^{2}-F_{m+6 n}^{2} \tag{*}
\end{equation*}
$$

In our solution to $B-576$, we establish that

Thus,

$$
A=5 F_{m+6 n+2} F_{m+6 n+1} .
$$

$$
4 A / 5=4 F_{m+6 n+2} F_{m+6 n+1} .
$$

But it is known that $4 F_{k} F_{k-1}=F_{k+1}^{2}-F_{k-2}^{2}$ [see $\left(I_{36}\right)$ on p. 59 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969], so (*) follows.

Also solved by Paul S. Bruckman, L.A. G. Dresel, Piero Filipponi, George Koutsoukellis, Andreas N. Philippou \& Demetris Antzoulakos, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

$$
\text { Zeckendorf Representation for }[\alpha F]
$$

B-578 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy
It is known (Zeckendorf's theorem) that every positive integer $N$ can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let $\alpha=(1+\sqrt{5}) / 2$ and $[x]$ denote the greatest integer not exceeding $x$. Denote by $f(N)$ the number of $F$-addends in the Zeckendorf representation for $N$. For positive integers $n$, prove that $f\left(\left[\alpha F_{n}\right]\right)=1$ if $n$ is odd.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI
It suffices to show that, for each positive integer $n,\left[\alpha F_{2 n-1}\right]$ is a Fibonacci number. We shall show that,
for each positive integer $n,\left[\alpha F_{2 n-1}\right]=F_{2 n}$.
Let $n$ be an arbitrary positive integer, and let $b=(1-\sqrt{5}) / 2$. It is known that, for each positive integer $k, a F_{k}=F_{k+1}-b^{k}$ [see p. 34 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. So $a F_{2 n-1}=F_{2 n}-b^{2 n-1}=F_{2 n}+(-b)^{2 n-1}$. Since $0<-b<1,0<(-b)^{2 n-1}<1$. It follows that $\left[a F_{2 n-1}\right]=F_{2 n}$.

Also solved by Paul S. Bruckman, L. Cseh, L.A. G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

> Zeckendorf Representation, Even Case

B-579 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy
Using the notation of $B-578$, prove that $f\left(\left[\alpha F_{n}\right]\right)=n / 2$ when $n$ is even.
Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI
Let $n$ be an arbitrary positive integer. We shall show that the Zeckendorf representation for $\left[\alpha F_{2 n}\right]$ is $F_{2}+F_{4}+F_{6}+\cdots+F_{2 n}$, which implies the required result.

Let $b=(1-\sqrt{5}) / 2$. It is known that

$$
a F_{2 n}=F_{2 n+1}-b^{2 n}
$$

[see p. 34 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. Since $0<b^{2}<1,0<b^{2 n}<1$. It follows that $\left[\alpha F_{2 n}\right]=F_{2 n+1}-1$.
But

$$
F_{2 n+1}-1=F_{2}+F_{4}+F_{6}+\cdots+F_{2 n}
$$

by ( $I_{6}$ ) (Ibid., p. 56). Hence, the Zeckendorf representation for $\left[a F_{2 n}\right]$ is

$$
F_{2}+F_{4}+F_{6}+\cdots+F_{2 n}
$$

completing our solution.
Also solved by Paul S. Bruckman, L. Cseh, L.A. G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

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