ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

and

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$$\begin{split} F_{n+2} &= F_{n+1} + F_n, \ F_0 &= 0, \ F_1 &= 1 \\ L_{n+2} &= L_{n+1} + L_n, \ L_0 &= 2, \ L_1 &= 1. \end{split}$$

PROBLEMS PROPOSED IN THIS ISSUE

B-598 Proposed by Herta T. Freitag, Roanoke VA

For which positive integers n is $(2L_n, L_{2n} - 3, L_{2n} - 1)$ a Pythagorean triple? For which of these n's is the triple primitive?

B-599 Proposed by Herta T. Freitag, Roanoke, VA

Do B-598 with the triple now $(2L_n, L_{2n} + 1, L_{2n} + 3)$.

B-600 Proposed by Philip L. Mana, Albuquerque, NM

Let *n* be any positive integer and $m = n^{13} - n$. Prove that F_n is an integral multiple of 30290.

B-601 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let $A_{n,k} = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k$. Find the smallest k in {2, 3, 4, ...} such that $A_{n,k}$ is an integer for every n in {0, 1, 2, ...}.

B-602 Proposed by Paul S. Bruckman, Fair Oaks, CA

Let H_n represent either F_n or L_n .

- (a) Find a simplified expression for $\frac{1}{H_n} \frac{1}{H_{n+1}} \frac{1}{H_{n+2}}$.
- (b) Use the result of (a) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + 2 \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}} \cdot$$

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B-603 Proposed by Paul S. Bruckman, Fair Oaks, CA

Do the Lucas analogue of B-602(b).

SOLUTIONS

Downrounded Square Roots

B-574 Proposed by Valentina Bakinova, Rondout Valley, NY

Let a_1, a_2, \ldots be defined by $a_1 = 1$ and $a_{n+1} = \lfloor \sqrt{s_n} \rfloor$, where $s_n = a_1 + a_2 + \cdots + a_n$ and $\lfloor x \rfloor$ is the integer with $x - 1 < \lfloor x \rfloor \le x$. Find $a_{100}, s_{100}, a_{1000}$, and s_{1000} .

Solution by L.A.G. Dressel, University of Reading, England

Starting with $s_1 = 1$, we have $a_2 = a_3 = a_4 = 1$ and $s_4 = 4$. Suppose now that, for some integer h, $h \ge 2$, we have $s_t = h^2$. Then, since $(h + 1)^2 = h^2 + 2h + 1$, we obtain

 $a_{t+1} = a_{t+2} = a_{t+3} = h$ and $s_{t+3} = (h+1)^2 + h - 1$; further,

 $a_{t+4} = a_{t+5} = h + 1$ and $s_{t+5} = (h + 2)^2 + h - 2$,

and continuing as long as $j \leq h$, $s_{t+2j+1} = (h+j)^2 + h - j$, so that for j = k we obtain $s_{t+2h+1} = (2h)^2$.

Since $s_4 = 2^2$, it follows that whenever s_n is a perfect square it is of the form 2^{2i} (i = 0, 1, 2, ...), and that if

 $s_{t_i} = 2^{2i}$ and $s_{t_{i+1}} = 2^{2(i+1)}$,

then $t_{i+1} = t_i + 2^{i+1} + 1$.

Since $s_1 = 1$, $t_0 = 1$, and we can show that

 $t_i = 2^{i+1} + i - 1$, for i = 0, 1, 2, ...

To find a_{100} and s_{100} : we have $t_5 = 64 + 4 = 68$, so that $s_{68} = (32)^2$,

$$s_{99} = (32 + 15)^2 + 32 - 15, a_{100} = 47, s_{100} = (47)^2 + 64 = 2273.$$

To find a_{1000} and s_{1000} : $t_8 = 2^9 + 7 = 519$ and $s_{519} = (256)^2$,

 $s_{998} = (256 + 239)^2 + 256 - 239, a_{999} = a_{1000} = 495$

and

$$s_{1000} = (256 + 240)^2 + 256 - 240 = (496)^2 + 16 = 246032.$$

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, J. Suck, M. Wachtel, and the proposer.

Summing Products

B-575 Proposed by L.A.G. Dresel, Reading, England

Let R_n and S_n be sequences defined by given values R_0 , R_1 , S_0 , S_1 and the recurrence relations $R_{n+1} = rR_n + tR_{n-1}$ and $S_{n+1} = sS_n + tS_{n-1}$, where r, s, t are constants and $n = 1, 2, 3, \ldots$. Show that

$$(n + s) \sum_{k=1}^{n} R_k S_k t^{n-k} = (R_{n+1} S_n + R_n S_{n+1}) - t^n (R_1 S_0 + R_0 S_1).$$
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Solution by J. Suck, Essen, Germany

This identity may be hard to dream up but is easy to prove by induction: For n = 1, the left-hand side is $(r + s)R_1S_1$, and the right-hand side is $(rR_1 + tR_0)S_1 + R_1(sS_1 + tS_0) - t(R_1S_0 + R_0S_1)$,

i.e., both are the same.

For the step from n to n + 1, we have to show that

 $t(R_{n+1}S_n + R_nS_{n+1}) + (r + s)R_{n+1}S_{n+1}$

 $= (rR_{n+1} + tR_n)S_{n+1} + R_{n+1}(sS_{n+1} + tS_n),$

which, after a little sorting, is seen to be true.

Also solved by Paul S. Bruckman, L. Cseh, Piero Filipponi & Adina Di Porto, L. Kuipers, Andreas N. Philippou & Demetris Antzoulakos, George Philippou, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Product of Three Fibonacci Numbers

B-576 Proposed by Herta T. Freitag, Roanoke, VA

Let $A = L_{2m+3(4n+1)} + (-1)^m$. Show that A is a product of three Fibonacci numbers for all positive integers m and n.

Solution by Lawrence Somer, Washington, D.C.

We prove the more general result that, if $r \ge 1$, then

 $L_{2r+1} + (-1)^{r+1} = 5F_rF_{r+1} = F_5F_rF_{r+1}.$

Note that, if 2r + 1 = 2m + 3(4n + 1), then

 $m \equiv r + 1 \pmod{2}$ and $(-1)^m = (-1)^{r+1}$.

By the Binet formulas and using the fact that $\alpha\beta = -1$, $5F_rF_{r+1} = 5[(\alpha^r - \beta^r)/\sqrt{5}][(\alpha^{r+1} - \beta^{r+1})/\sqrt{5}]$ $= \alpha^{2r+1} + \beta^{2r+1} - (\alpha\beta)^r(\alpha + \beta)$ $= L_{2r+1} - (-1)^rL_1 = L_{2r+1} + (-1)^{r+1}$,

and we are done.

Also solved by Paul S. Bruckman, L.A.G. Dresel, Piero Filipponi, George Koutsoukellis, L. Kuipers, Andreas N. Philippou & Demetris Antzoulakos, Bob Prielipp, H.-J. Seiffert, Sahib Singh, J. Suck, and the proposer.

Difference of Squares

B-577 Proposed by Herta T. Freitag, Roanoke, VA

Let A be as in B-575. Show that 4A/5 is a difference of squares of Fibonacci numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Let m and n be arbitrary positive integers. We shall show that

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$$4A/5 = F_{m+6n+3}^2 - F_{m+6n}^2 \cdot$$

In our solution to B-576, we establish that

Thus,

 $4A/5 = 4F_{m+6n+2}F_{m+6n+1}.$

 $A = 5F_{m+6n+2}F_{m+6n+1}$

But it is known that $4F_kF_{k-1} = F_{k+1}^2 - F_{k-2}^2$ [see (I₃₆) on p. 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969], so (*) follows.

Also solved by Paul S. Bruckman, L.A.G. Dresel, Piero Filipponi, George Koutsoukellis, Andreas N. Philippou & Demetris Antzoulakos, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Zeckendorf Representation for [aF]

B-578 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

It is known (Zeckendorf's theorem) that every positive integer N can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let $a = (1 + \sqrt{5})/2$ and [x] denote the greatest integer not exceeding x. Denote by f(N) the number of F-addends in the Zeckendorf representation for N. For positive integers n, prove that $f([aF_n]) = 1$ if n is odd.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

It suffices to show that, for each positive integer n, $[aF_{2n-1}]$ is a Fibonacci number. We shall show that,

for each positive integer n, $[aF_{2n-1}] = F_{2n}$.

Let *n* be an arbitrary positive integer, and let $b = (1 - \sqrt{5})/2$. It is known that, for each positive integer *k*, $aF_k = F_{k+1} - b^k$ [see p. 34 of *Fibonacci* and *Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. So $aF_{2n-1} = F_{2n} - b^{2n-1} = F_{2n} + (-b)^{2n-1}$. Since 0 < -b < 1, $0 < (-b)^{2n-1} < 1$. It follows that $[aF_{2n-1}] = F_{2n}$.

Also solved by Paul S. Bruckman, L. Cseh, L.A.G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Zeckendorf Representation, Even Case

B-579 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

Using the notation of B-578, prove that $f([aF_n]) = n/2$ when n is even.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Let *n* be an arbitrary positive integer. We shall show that the Zeckendorf representation for $[aF_{2n}]$ is $F_2 + F_4 + F_6 + \cdots + F_{2n}$, which implies the required result.

Let $b = (1 - \sqrt{5})/2$. It is known that

 $aF_{2n} = F_{2n+1} - b^{2n}$

(*)

[see p. 34 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969]. Since $0 < b^2 < 1$, $0 < b^{2n} < 1$. It follows that

But

 $F_{2n+1} - 1 = F_2 + F_4 + F_6 + \cdots + F_{2n}$

by (I₆) (Ibid., p. 56). Hence, the Zeckendorf representation for $[aF_{2n}]$ is $F_2 + F_4 + F_6 + \cdots + F_{2n}$

completing our solution.

 $[aF_{2n}] = F_{2n+1} - 1.$

Also solved by Paul S. Bruckman, L. Cseh, L.A.G. Dresel, Herta T. Freitag, L. Kuipers, Imre Merenyi, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

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