A NOTE ON $n(x, y)$-REFLECTED LATTICE PATHS

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INTRODUCTION

A natural bijection between the class of lattice paths from (0, 0) to ($2m$, $2m$) having the property that, for each (x, y) in the path, ($2m - x$, $2m - y$) is also on the path and the class of partitions of $2m^2$ into at most $2m$ parts, each part $\leqslant 2m$ and the parts which are strictly less than $2m$ can be paired such that the sum of each pair is $2m$, is shown.

1. DEFINITION AND THE MAIN RESULT

Describing the n-reflected lattice paths [paths from (0, 0) to (n, n) having the property that, for each (x, y) in the path, ($n - y$, $n - x$) is also on the path] of the paper "Hook Differences and Lattice Paths" [1] as $n(y, x)$-reflected, we define here $n(x, y)$-reflected lattice paths as follows:

Definition: A lattice path from (0, 0) to (n, n) is said to be $n(x, y)$-reflected if, for each (x, y) in the path, ($n - x$, $n - y$) is also on the path.

Example: The two $2(x, y)$-reflected lattice paths are:

In the present note we propose to prove the following.

Theorem: The number of partitions of $2m^2$ into at most $2m$ parts each $\leqslant 2m$ and the parts which are strictly less than $2m$ can be paired such that the sum of each pair is $2m$ equals $\binom{2m}{m}$.

2. PROOF OF THE THEOREM

We describe a partition of $2m^2$ as a multiset

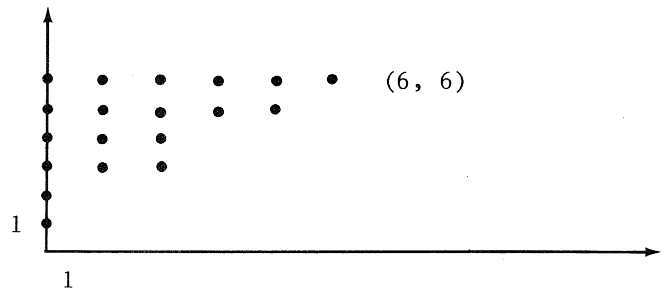
$$\mu = \mu(m) := [a_1, \ldots, a_s]$$

of $s(\in\{1, 2, \ldots, 2m^2\})$ positive integers a_i ($i = 1, 2, \ldots, s$) such that

$$\sum_{i=1}^{s} a_i = 2m^2 \quad (\text{conventionally, } a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s).$$

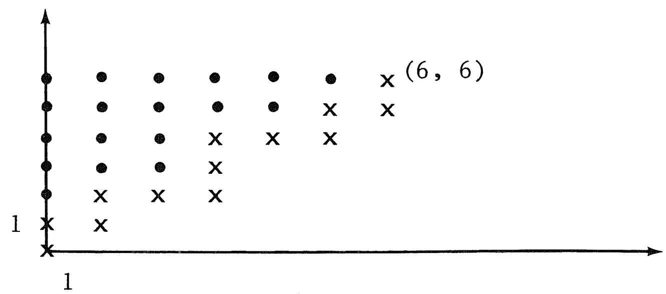
In this notation let $\mathcal{S}(m)$ denote the set of all partitions $\mu = [a_1, a_2, \ldots, a_s]$ of $2m^2$ such that $s \leqslant 2m$, $2m \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s$; and, all of the a_j for which $a_j < 2m$ can be paired such that the sum of each pair equals $2m$. Further, let $\mathfrak{I}(m)$ denote the set of all $2m(x, y)$-reflected lattice paths. To establish a one-to-one correspondence from $\mathcal{S}(m)$ onto $\mathfrak{I}(m)$, we represent any $\mu = [a_1, a_2, \ldots, a_s] \in \mathcal{S}(m)$ by its Ferrers graph in the coordinate plane as follows:

We fit the leftmost node of the i^{th} row of nodes (counted by a_i) over the point $(0, 2m - i + 1)$ as shown in Graph A (in the graph, $m = 3$ and $\mu = [6, 5, 3, 3, 1]$).



Graph A

We now place crosses at one unit of length below every free horizontal node and at one unit of length to the right of every free vertical node. Through these crosses, we then complete the lattice path from (0, 0) to $(2m, 2m)$, as shown in Graph B.



Graph B

We observe that each partition μ corresponds uniquely to a $2m(x, y)$-reflected lattice path. It may be noted here that the corresponding path will not be $2m(x, y)$-reflected if

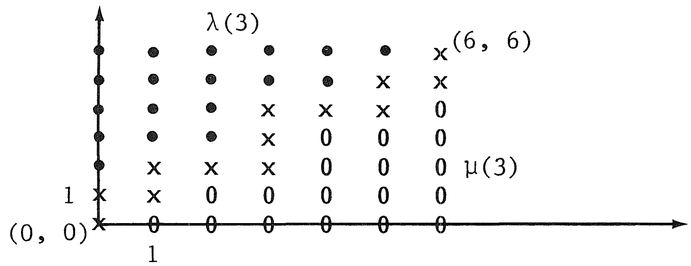
$$s = 2m = a_1. \tag{1}$$

For, in this case, $(2m, 2m - 1)$ belongs to the path, but $(0, 1) = (2m - 2m, 2m - (2m - 1))$ does not. Therefore, in order to prove that the correspondence is one-to-one and onto, we first rule out the possibility (1) under the conditions of the theorem. There are only three possible cases: (i) $a_1 > a_2$. In this case, if (1) is true, then there are $2m - 1$ parts, viz. $a_2, \ldots, a_s$, that are strictly less than $2m$. Being odd in number, these parts cannot be paired; hence, (1) is false. (ii) $a_1 = a_2 = \cdots = a_r$, where r ($\geqslant 1$) is odd. In this case, if (1) is true, then the number of parts that are $< 2m$ is $2m - r$. Again, since $2m - r$ is odd, the parts that are $< 2m$ cannot be paired; hence, (1) is not possible. (iii) $a_1 = a_2 = \cdots = a_r$, where r ($\geqslant 2$) is even. As in the previous case, if (1) is true, then the number of parts that are $< 2m$ is $2m - r$. But in this case, $2m - r$ is even. So the parts that are $< 2m$ can be paired. However, since the sum of each pair is $2m$, the number being partitioned is:

$$2m \cdot r + \frac{(2m - r)}{2} \cdot 2m = 2m^2 + mr > 2m^2.$$

This is a contradiction since we are considering the partitions of $2m^2$. Thus, (1) does not hold true.

We also note that each $2m(x, y)$-reflected lattice path uniquely splits $4m^2$ into two identical partitions of $2m^2$, say, $\lambda(m)$ and $\mu(m)$. (See Graph C, where $m = 3$ and $\lambda(3) = [6, 5, 3, 3, 1]$).



Graph C

Now if a_i $(i = 1, 2, \ldots, s) \in \lambda$, and $a_i < 2m$, there must exist b_j $(j = 1, 2, \ldots, s) \in \mu$, where $b_j < 2m$, such that $a_i + b_j = 2m$. But since λ and μ are identical, $b_j = a_k$ for some $k \in \{1, 2, \ldots, s\}$. Thus, $a_i + a_k = 2m$. This is how the restriction "all of the a_j for which $a_j < 2m$ can be paired such that the sum of each pair equals $2m$" enters into the argument. After establishing a one-to-one correspondence from $\mathcal{S}(m)$ onto $\mathcal{J}(m)$, we use the fact that each $2m(x, y)$-reflected lattice path determines and is determined uniquely by its first half, i.e., the nondecreasing path between (0, 0) to (m, m). Hence, the number of $2m(x, y)$-reflected lattice paths or the number of relevant partitions equals the number of paths between (0, 0) to (m, m), i.e., $\binom{2m}{m}$. This completes the proof of the theorem.

As an example, let us consider the case in which $m = 3$. We get the following relevant partitions:

$$3^6,\ 43^42,\ 4^23^22^2,\ 4^32^3,\ 53^41,\ 54^22^21,\ 543^221,\ 5^23^21^2,\ 5^2421^2,\ 5^31^3,$$
$$63^4,\ 64^22^2,\ 643^22,\ 653^21,\ 65^21^2,\ 6^23^2,\ 6^242,\ 6^251,\ 6^3,\ 65421.$$

We remark here that in all there are 58 partitions of 18 into at most 6 parts and each part $\leqslant 6$ $\left(\text{see [2], p. 243, coefficient of } q^{18} \text{ in the expansion of } \begin{bmatrix} 12 \\ 8 \end{bmatrix}\right)$. But 38 partitions, such as 6543, 5^33, 543^3, 4^33^2, 6^22^3, 5^321, etc., do not satisfy the condition "the parts which are < 6 can be paired such that the sum of each pair is 6."

ACKNOWLEDGMENT

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REFERENCES

1. A. K. Agarwal & G. E. Andrews. "Hook Differences and Lattice Paths." *Journal of Statistical Planning and Inference* (to appear).
2. G. E. Andrews. "The Theory of Partitions." In *Encyclopedia of Mathematics and Its Applications*. New York: Addison-Wesley, 1976.

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