A GENERALIZATION OF FIBONACCI POLYNOMIALS AND A REPRESENTATION OF GEGENBAUER POLYNOMIALS OF INTEGER ORDER

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(Submitted September 1985)

1. INTRODUCTION

Various sequences of polynomials by the name of Fibonacci and Lucas polynomials occur in the literature. For example, Doman & Williams [2] introduced the polynomials

$$F_{n+1}(z) := \sum_{m=0}^{[n/2]} \binom{n-m}{m} z^m, \tag{1}$$

$$L_n(z) := \sum_{m=0}^{[n/2]} \frac{n}{n-m}\binom{n-m}{m} z^m, \tag{2}$$

for $n = 1, 2, 3, \ldots$, and $F_0(z) := 0$, $F_1(z) := 1$, $L_0(z) := 2$; $[n/2]$ denotes the integer part of $n/2$. Several properties of these polynomials were derived in [2] and, more recently, by Galvez & Dehesa [3].

The Fibonacci and Lucas polynomials which occur, for example, in [4], are different from but closely related to the $F_n(z)$ and $L_n(z)$. The properties derived in [4] and in the papers cited there can easily be adapted to the polynomials defined in (1) and (2); they mainly concern zeros and divisibility properties.

In [2], the connection to the Gegenbauer (or ultraspherical) and Chebyshev polynomials $C_n^{\alpha}(z)$ and $T_n(z)$ was given, namely

$$C_n^1(z) = (2z)^n F_{n+1}(-1/4z^2),$$

$$T_n(z) = \frac{1}{2}(2z)^n L_n(-1/4z^2).$$

We also note that $C_n^1(z) = U_n(z)$, the Chebyshev polynomial of the second kind. Because $2T_n(z) = nC_n^0(z)$ (see, e.g., [1], p. 779), we now have

$$F_{n+1}(z) = (-z)^{n/2}\, C_n^1(1/2\sqrt{-z}), \tag{3}$$

$$\frac{1}{n} L_n(z) = (-z)^{n/2}\, C_n^0(1/2\sqrt{-z}); \tag{4}$$

here and in the following the square root is to be considered as the principal branch.

The purpose of this note is to use these identities as a starting point to define a wider class of sequences of polynomials which contains (1) and (2) as special cases, and to derive some properties.

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*Supported by a Killam Postdoctoral Fellowship.

2. THE POLYNOMIALS $F_n^{(k)}(z)$

For $k = -1, 0, 1, \ldots,$ we introduce

$$F_n^{(k)}(z) := (-z)^{n/2}\, C_n^{k+1}(1/2\sqrt{-z}); \tag{5}$$

by (3) and (4), we have the special cases

$$F_n^{(0)}(z) = F_{n+1}(z) \quad \text{and} \quad F_n^{(-1)}(z) = L_n(z)/n.$$

We now use the explicit expressions for the Gegenbauer polynomials (see, e.g., [1], p. 775):

$$C_n^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(\alpha + n - m)}{m!\,(n-2m)!}(2x)^{n-2m}, \tag{6}$$

for $\alpha > -1/2$, $\alpha \neq 0$, and

$$C_n^{0}(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m}. \tag{7}$$

The connection between (7) and (2) is immediate and, for $\alpha = k + 1 \geqslant 1$, we have

$$\frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha + n - m)}{m!\,(n-2m)!} = \frac{(n+k-m)!}{k!\,m!\,(n-2m)!} = \binom{n+k-m}{m}\binom{n+k-2m}{k}$$

with (6) and (5), this yields the explicit expression

$$F_n^{(k)}(z) = \sum_{m=0}^{[n/2]} \binom{n+k-m}{m}\binom{n+k-2m}{k} z^m, \tag{8}$$

for $k \geqslant 0$. This could also serve as a definition of the $F_n^{(k)}(z)$, in analogy to (1).

3. SOME PROPERTIES

With (5) and the recurrence relation for Gegenbauer polynomials (see, e.g., [1], p. 782), we obtain

$$(n+1)F_{n+1}^{(k)}(z) = (n+k+1)F_n^{(k)}(z) + (n+2k+1)zF_{n-1}^{(k)}(z). \tag{9}$$

More properties of the $F_n^{(k)}(z)$ can be derived, with (5), from the corresponding properties of the Gegenbauer polynomials. This includes generating functions, differential relations, and more recurrence relations; we just mention

$$\frac{d}{dz} F_{n+1}^{(k)}(z) = (k+1)F_{n-1}^{(k+1)}(z) \qquad (\text{for } k \geqslant 0),$$

and

$$\frac{d}{dz} L_n(z) = nF_{n-1}(z), \tag{10}$$

which can also be verified directly using (8), (1), and (2). If we differentiate the recurrence

$$P_{n+1}(z) = P_n(z) + zP_{n-1}(z) \tag{11}$$

which, by (9), holds for $L_n(z)$ and $F_n(z)$, we get, with (10),

$$(n+1)F_n(z) = nF_{n-1}(z) + L_{n-1}(z) + (n-1)zF_{n-2}(z);$$

this, combined with (11), for $F_n(z)$, yields

$$L_{n-1}(z) = 2F_n(z) - F_{n-1}(z).$$

This last equation can also be derived from the corresponding well-known identity connecting the Chebyshev polynomials of the first and second kind.

The following recurrence relation involves polynomials $F_n^{(k)}(z)$ of different orders $k \geqslant 1$.

$$F_{n+2}^{(k)}(z) - F_{n+1}^{(k)}(z) - zF_n^{(k)}(z) = F_{n+2}^{(k-1)}(z),$$

which can be verified by elementary manipulations, using (8).

4. THE $F_n^{(k)}(z)$ AS ELEMENTARY SYMMETRIC FUNCTIONS

We begin with the following

Lemma: (a) For integers $n \geqslant 0$ and for complex $z \neq 1$ and x, we have

$$\sum_{j=0}^{n} (-1)^j F_j^{(n-j)}(x) z^{n-j} = (z-1)^n F_{n+1}\left(\frac{x}{(z-1)^2}\right) \tag{12}$$

(b) $$\sum_{j=0}^{n} (-1)^j F_j^{(n-j)}(x) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ x^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof: Let $f_n(x, z)$ denote the left-hand side of (12). With (8), we have

$$\begin{aligned} f_n(x, z) &= \sum_{j=0}^{n} (-1)^j \sum_{m=0}^{[j/2]} \binom{n-m}{m}\binom{n-2m}{n-j} x^m z^{n-j} \\ &= \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} \sum_{j=2m}^{n} (-1) \binom{n-2m}{j-2m} z^{n-j} \\ &= \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} \sum_{j=0}^{n-2m} (-1)^j \binom{n-2m}{j} z^{n-2m-j}, \end{aligned}$$

which yields assertion (b) if we put $z = 1$. For $z \neq 1$, we have

$$f_n(x, z) = \sum_{m=0}^{[n/2]} x^m \binom{n-m}{m} (z-1)^{n-2m} = (z-1)^n \sum_{m=0}^{[n/2]} \binom{n-m}{m} \left(\frac{x}{(z-1)^2}\right)^m,$$

which proves (a).

Proposition: For $k = 1, 2, \ldots, n$, we have

$$F_k^{(n-k)}(x) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} A_{j_1}^{(n)}(x) \ldots A_{j_k}^{(n)}(x),$$

where

$$A_j^{(n)}(x) := 1 + 2\sqrt{-x} \cos \frac{j\pi}{n+1}.$$

Proof: Because $C_n^1(z) = U_n(z)$, we have, with (3) and the definition of $A_j^{(n)}(x)$,

$$F_{n+1}(x(A_j^{(n)}(x) - 1)^{-2}) = F_{n+1}\left(-1/4 \cos^2 \frac{j\pi}{n+1}\right)$$

$$= \left(2 \cos \frac{j\pi}{n+1}\right)^{-n} U_n\left(\cos \frac{j\pi}{n+1}\right).$$

Now $\cos(j\pi/(n+1))$, for $j = 1, 2, \ldots, n$, are known to be the zeros of the Chebyshev polynomials of the second kind $U_n(z)$. Furthermore, if n is odd, then $\cos(j\pi/(n+1)) = 0$ for $j = (n+1)/2$, in which case $A_j^{(n)}(x) = 1$ for all x. So we have, by both parts of the Lemma,

$$\sum_{k=0}^{n} (-1)^k F_k^{(n-k)}(x)(A_j^{(n)}(x))^{n-k} = 0$$

for all $j = 1, 2, \ldots, n$. But this means that the $F_k^{(n-k)}(x)$, $k = 0, 1, \ldots, n$, with x held constant, are the elementary symmetric functions of the n roots $A_j^{(n)}(x)$ of $f(x, z) = 0$. This proves the Proposition.

Finally, if we let $x = 1/2\sqrt{-z}$, the proposition together with (5) yields the following representation of the ultraspherical polynomials of integer order.

Corollary: If $k \geqslant 1$ is an integer, then

$$C_n^k(x) = 2^n \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n+k-1} \left(x + \cos \frac{j_1\pi}{n+k}\right) \cdot \cdots \cdot \left(x + \cos \frac{j_n\pi}{n+k}\right).$$

In closing, we note that [5] and [6] deal with Gegenbauer polynomials from another (related) point of view.

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