HOGGATT SEQUENCES AND LEXICOGRAPHIC ORDERING

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DEDICATION

While I was a graduate student at San Jose State University, Vern Hoggatt and I worked with sequences of positive integers which we were calling "generalized r-nacci numbers." In this paper, I have gathered some of our results concerning these sequences which I have renamed "the Hoggatt sequences." I would like to dedicate this paper to the memory of Professor Hoggatt.

M. A. O.

INTRODUCTION

The Zeckendorf Theorem states that every positive integer can be represented as a sum of distinct Fibonacci numbers and that this representation is unique, provided no two consecutive Fibonacci numbers appear in any sum.

In [2] the Zeckendorf Theorem is extended to a class of sequences obtained from the generalized Fibonacci polynomials; in particular, an analogous theorem holds for the generalized Fibonacci sequences. In Section 1, a collection of sequences called the Hoggatt sequences is introduced, and it is shown that these sequences also enjoy a "Zeckendorf Theorem"; in fact, the Hoggatt sequences share many of the representation and ordering properties of the generalized Fibonacci sequences discussed in [2] and [3].

1. HOGGATT SEQUENCES AND ZECKENDORF REPRESENTATIONS

For each fixed integer r with $r \geqslant 2$, the generalized Fibonacci polynomials yield a generalized Fibonacci sequence [2] which will be denoted $\{R_n\}_{n=1}^{\infty}$. The *generalized Fibonacci sequence* associated with the integer r can be defined as follows [3]:

$$R_1 = 1;$$

$$R_j = 2^{j-2} \text{ for } j = 2, 3, \ldots, r;$$

$$R_{k+r} = R_{k+r-1} + R_{k+r-2} + \cdots + R_k \text{ for all positive integers } k.$$

Note that with $r = 2, 3, 4,$ and 5 we obtain, respectively, the Fibonacci numbers $\{F_n\}$, the Tribonacci numbers $\{T_n\}$, the Quadranacci numbers $\{Q_n\}$, and the Pentanacci numbers $\{P_n\}$.

The *Hoggatt sequence of degree* r, where r is once again a fixed integer greater than 1, will be denoted $\{\mathcal{R}_n\}$ and can be obtained by taking differences of adjacent generalized Fibonacci numbers; more precisely, $\mathcal{R}_n = R_{n+2} - R_{n+1}$ for all positive integers n. The defining properties of the sequences $\{R_n\}$ and $\{\mathcal{R}_n\}$ give rise to the following recursive description of the Hoggatt sequence of degree r:

$$\mathcal{R}_j = 2^{j-1} \text{ for } j = 1, 2, \ldots, r - 1;$$
$$\mathcal{R}_r = 2^{r-1} - 1 = \mathcal{R}_1 + \mathcal{R}_2 + \cdots + \mathcal{R}_{r-1};$$
$$\mathcal{R}_{k+r} = \mathcal{R}_{k+r-1} + \mathcal{R}_{k+r-2} + \cdots + \mathcal{R}_k \text{ for all positive integers } k.$$

Note that the second-degree Hoggatt sequence coincides with the Fibonacci sequence; moreover, for $r > 2$, the sequences $\{R_n\}$ and $\{\mathcal{R}_n\}$ differ in their initial (and subsequent) entries but share a common recursion relation.

Identities similar (but not identical) to those developed for the generalized Fibonacci sequences in [3] can be obtained for the Hoggatt sequences.

For $r = 2$ the Hoggatt sequence is the Fibonacci sequence $\{F_n\}$, and we have the two identities

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$$

and

$$F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2} - 1.$$

Let the third-degree Hoggatt sequence be denoted $\{\mathcal{T}_n\}$. Three identities arise in this case:

$$(\mathcal{T}_2 + \mathcal{T}_3) + (\mathcal{T}_5 + \mathcal{T}_6) + \cdots (\mathcal{T}_{3n-1} + \mathcal{T}_{3n}) = \mathcal{T}_{3n+1} - 1;$$
$$\mathcal{T}_1 + (\mathcal{T}_3 + \mathcal{T}_4) + (\mathcal{T}_6 + \mathcal{T}_7) + \cdots + (\mathcal{T}_{3n} + \mathcal{T}_{3n+1}) = \mathcal{T}_{3n+2} - 1;$$
$$\mathcal{T}_2 + (\mathcal{T}_4 + \mathcal{T}_5) + (\mathcal{T}_7 + \mathcal{T}_8) + \cdots + (\mathcal{T}_{3n+1} + \mathcal{T}_{3n+2}) = \mathcal{T}_{3n+3} - 1.$$

In general, we have the following lemma.

Lemma 1.1: For each integer r greater than 1, there arise r identities involving groupings of $(r-1)$ consecutive terms of the Hoggatt sequence of degree r.

$$\begin{aligned}
&(\mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_r) + (\mathcal{R}_{r+2} + \mathcal{R}_{r+3} + \cdots + \mathcal{R}_{2r}) + \cdots \\
&\quad + (\mathcal{R}_{r(n-1)+2} + \mathcal{R}_{r(n-1)+3} + \cdots + \mathcal{R}_{rn}) = \mathcal{R}_{rn+1} - 1; \\
&\mathcal{R}_1 + (\mathcal{R}_3 + \mathcal{R}_4 + \cdots + \mathcal{R}_{r+1}) + (\mathcal{R}_{r+3} + \mathcal{R}_{r+4} + \cdots + \mathcal{R}_{2r+1}) + \cdots \\
&\quad + (\mathcal{R}_{r(n-1)+3} + \mathcal{R}_{r(n-1)+4} + \cdots + \mathcal{R}_{rn+1}) = \mathcal{R}_{rn+2} - 1; \\
&\mathcal{R}_1 + \mathcal{R}_2 + (\mathcal{R}_4 + \mathcal{R}_5 + \cdots + \mathcal{R}_{r+2}) + (\mathcal{R}_{r+4} + \mathcal{R}_{r+5} + \cdots + \mathcal{R}_{2r+2}) + \cdots \\
&\quad + (\mathcal{R}_{r(n-1)+4} + \mathcal{R}_{r(n-1)+5} + \cdots + \mathcal{R}_{rn+2}) = \mathcal{R}_{rn+3} - 1; \\
&\quad \vdots \\
&\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_{r-2} + (\mathcal{R}_r + \mathcal{R}_{r+1} + \cdots + \mathcal{R}_{2r-2}) + \cdots \\
&\quad + (\mathcal{R}_{rn} + \mathcal{R}_{rn+1} + \cdots + \mathcal{R}_{rn+r-2}) = \mathcal{R}_{rn+r-1} - 1; \\
&\mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 + \cdots + \mathcal{R}_{r-1} + (\mathcal{R}_{r+1} + \mathcal{R}_{r+2} + \cdots + \mathcal{R}_{2r-1}) + \cdots \\
&\quad + (\mathcal{R}_{rn+1} + \mathcal{R}_{rn+2} + \cdots + \mathcal{R}_{rn+r-1}) = \mathcal{R}_{rn+r} - 1.
\end{aligned}$$

Proof: For a fixed r, each of the identities can be verified by adding 1 to the expression on the left and applying the appropriate recursion relation.

In the first equation, note that

$$1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_r = \mathcal{R}_{r+1}.$$

When the term $\mathcal{R}_{r+1}$ is added to the next $(r-1)$ consecutive terms the result is $\mathcal{R}_{2r+1}$, which can be added to the next $(r-1)$ consecutive terms; this process can be repeated until addition yields $\mathcal{R}_{rn+1}$.

In general for the i^{th} equation, where $2 \leqslant i \leqslant r - 1$, note that

$$1 + \mathcal{R}_1 + \mathcal{R}_2 + \cdots + \mathcal{R}_{i-1} = 1 + 1 + 2 + \cdots + 2^{i-2} = 2^{i-1} = \mathcal{R}_i.$$

Since the next parenthetic expression is

$$\mathcal{R}_{i+1} + \mathcal{R}_{i+2} + \cdots + \mathcal{R}_{r+i-1},$$

the addition process described for the first equation can now be applied.

The final identity follows by recalling that $1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots + \mathcal{R}_{r-1} = \mathcal{R}_r$ and applying the addition process.

In [1] a proof of a Zeckendorf Theorem for the generalized Fibonacci polynomials is given; a consequence of this theorem is the existence and uniqueness of the Zeckendorf representation for positive integers in terms of the generalized Fibonacci numbers. A generalized Zeckendorf Theorem also holds for the Hoggatt numbers of degree r. That is, for a given r, every positive integer can be represented as the sum of distinct terms of the sequence $\{\mathcal{R}_n\}$ provided no r consecutive terms of the sequence are used in the representation; however, since the sum of the first $(r-1)$ terms of the sequence is $\mathcal{R}_r$, in order to ensure uniqueness of the representation, we must also require that no representation use the first $(r-1)$ consecutive terms of $\{\mathcal{R}_n\}$.

Theorem 1.2: For each fixed integer $r \geqslant 2$, every positive integer N has a unique representation in terms of $\{\mathcal{R}_n\}$ of the form

$$N = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_i\mathcal{R}_i, \text{ where } N_j \in \{0, 1\} \text{ for } j = 1, 2, \ldots, i,$$

$$N_1 N_2 \cdot \cdots \cdot N_{r-1} = 0,$$

and

$$N_k N_{k+1} \cdot \cdots \cdot N_{k+r-1} = 0 \text{ for all positive integers } k;$$

i.e., every integer has a unique Zeckendorf representation in terms of $\{\mathcal{R}_n\}$.

Proof: Note that for $r = 2$, the Hoggatt sequence in question is the Fibonacci sequence and the Zeckendorf Theorem holds.

The nature of the inductive proof of the theorem can best be seen by considering a particular small value of r. We concentrate our efforts on the case in which $r = 3$. Suppose for some n every positive integer $N \leqslant \mathcal{T}_{3n+2} - 1$ has a unique Zeckendorf representation; it suffices to prove that every positive integer $N \leqslant \mathcal{T}_{3n+3} - 1$ has a unique Zeckendorf representation.

It follows from Lemma 1.1 that

$$\mathcal{T}_{3n+2} - 1 = \mathcal{T}_1 + (\mathcal{T}_3 + \mathcal{T}_4) + (\mathcal{T}_6 + \mathcal{T}_7) + \cdots + (\mathcal{T}_{3n} + \mathcal{T}_{3n+1}),$$

and this equation must give the unique Zeckendorf representation for $\mathcal{T}_{3n+2} - 1$. Next, we note that the representation for $\mathcal{T}_{3n+2} - 1$ implies that the largest integer which can be represented without using $\mathcal{T}_{3n+2}$ or any succeeding term of $\{\mathcal{T}_n\}$ is $\mathcal{T}_{3n+2} - 1$; therefore, the term $\mathcal{T}_{3n+2}$ is itself the unique Zeckendorf representation for $\mathcal{T}_{3n+2}$.

Since $\mathcal{T}_{3n+1} - 1 < \mathcal{T}_{3n+2} - 1$, the integer $\mathcal{T}_{3n+1} - 1$ has a unique Zeckendorf representation. Moreover, this unique representation is given by the following identity from Lemma 1.1:

$$\mathcal{T}_{3n+1} - 1 = (\mathcal{T}_2 + \mathcal{T}_3) + (\mathcal{T}_5 + \mathcal{T}_6) + \cdots + (\mathcal{T}_{3n-1} + \mathcal{T}_{3n}).$$

An immediate consequence of the preceding observations is that

$$\mathcal{T}_{3n+2} + \mathcal{T}_{3n+1} - 1$$

is uniquely representable by

$$\mathfrak{J}_{3n+2} + (\mathfrak{J}_2 + \mathfrak{J}_3) + (\mathfrak{J}_5 + \mathfrak{J}_6) + \cdots + (\mathfrak{J}_{3n-1} + \mathfrak{J}_{3n}).$$

It also follows that, for any positive integer M less than $\mathfrak{J}_{3n+1}$, there is a unique Zeckendorf representation for $\mathfrak{J}_{3n+2} + M$ consisting of adding $\mathfrak{J}_{3n+2}$ to the unique Zeckendorf representation for M.

Finally, we apply the only remaining third-degree identity in Lemma 1.1. Since $\mathfrak{J}_{3n} - 1 < \mathfrak{J}_{3n+2} - 1$, the integer $\mathfrak{J}_{3n} - 1$ has a unique Zeckendorf representation, and this representation is given by the identity

$$\mathfrak{J}_{3n} - 1 = \mathfrak{J}_2 + (\mathfrak{J}_4 + \mathfrak{J}_5) + (\mathfrak{J}_7 + \mathfrak{J}_8) + \cdots + (\mathfrak{J}_{3n-2} + \mathfrak{J}_{3n-1}).$$

It follows immediately that

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + \mathfrak{J}_{3n} - 1$$

has the unique Zeckendorf representation

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + [\mathfrak{J}_2 + (\mathfrak{J}_4 + \mathfrak{J}_5) + (\mathfrak{J}_7 + \mathfrak{J}_8) + \cdots + (\mathfrak{J}_{3n-2} + \mathfrak{J}_{3n-1})].$$

It is also apparent that $\mathfrak{J}_{3n+2} + M$ has a unique Zeckendorf representation for every positive integer M less than $\mathfrak{J}_{3n+1} + \mathfrak{J}_{3n}$.

Noting that

$$\mathfrak{J}_{3n+2} + \mathfrak{J}_{3n+1} + \mathfrak{J}_{3n} - 1 = \mathfrak{J}_{3n+3} - 1$$

concludes the proof of the theorem in the case $r = 3$.

The only major difference between the proof for $r = 3$ and the proof for an arbitrary value of r is that in the general case all r identities appearing in Lemma 1.1 must be used.

2. THE HOGGATT SEQUENCE OF DEGREE 3

If $r = 3$, the associated Hoggatt sequence $\{\mathfrak{J}_n\}$ is defined by taking

$$\mathfrak{J}_1 = 1,\ \mathfrak{J}_2 = 2,\ \mathfrak{J}_3 = \mathfrak{J}_1 + \mathfrak{J}_2 = 1 + 2 = 3$$

and

$$\mathfrak{J}_i = \mathfrak{J}_{i-1} + \mathfrak{J}_{i-2} + \mathfrak{J}_{i-3} \text{ for } i \geqslant 4;$$

the first seven terms of the resulting sequence are:

$\mathfrak{J}_1$	$\mathfrak{J}_2$	$\mathfrak{J}_3$	$\mathfrak{J}_4$	$\mathfrak{J}_5$	$\mathfrak{J}_6$	$\mathfrak{J}_7$
1	2	3	6	11	20	37

By Theorem 1.2, every positive integer has a unique Zeckendorf representation in terms of the third-degree Hoggatt numbers. In the next theorem, we prove that the terms used in the Zeckendorf representation of integers give information about the natural ordering of the integers being represented; in particular, we investigate lexicographic orderings which were defined and examined in [3] and [5]. We now define this kind of ordering as in [3].

Let the positive integers be represented in terms of a strictly increasing sequence of integers, $\{A_n\}$, so that for integers M and N,

$$M = \sum_{i=1}^{k} M_i A_i \quad \text{and} \quad N = \sum_{i=1}^{k} N_i A_i,$$

where the coefficients M_i and N_i lie in the set $\{0, 1, 2, \ldots, q\}$ for some fixed integer q; moreover, suppose m is an integer such that $M_i = N_i$ for all $i > m$.

If, for every pair of integers M and N, $M_m > N_m$ implies $M > N$, then the representation is a *lexicographic ordering*.

In [3], identities analogous to those in Lemma 1.1 are used to show that the Zeckendorf representation of the positive integers in terms of the Tribonacci numbers is a lexicographic ordering; a similar proof is used in the following theorem.

Theorem 2.1: The Zeckendorf representation of the positive integers in terms of the third-degree Hoggatt sequence $\{\mathfrak{I}_n\}$ is a lexicographic ordering.

Proof: Let M and N be two positive integers expressed in Zeckendorf form in terms of the third-degree Hoggatt numbers; then, for some positive integer t,

$$M = \sum_{i=1}^{t} M_i \mathfrak{I}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i \mathfrak{I}_i,$$

where $M_i,\ N_i \in \{0,\ 1\}$, $M_1 M_2 = N_1 N_2 = 0$ and, for all i,

$$M_i M_{i+1} M_{i+2} = N_i N_{i+1} N_{i+2} = 0.$$

Let m be a positive integer such that $M_i = N_i$ for all $i > m$, and suppose that $M_m > N_m$. Then $M_m = 1$ and $N_m = 0$. In order to prove that $M > N$, we consider the following truncated portions of M and N:

$$M^* = M_1 \mathfrak{I}_1 + M_2 \mathfrak{I}_2 + \cdots + M_{m-1} \mathfrak{I}_{m-1} + \mathfrak{I}_m \geqslant \mathfrak{I}_m$$

and

$$N^* = N_1 \mathfrak{I}_1 + N_2 \mathfrak{I}_2 + \cdots + N_{m-1} \mathfrak{I}_{m-1}.$$

It is clear from the nature of the Zeckendorf representation and the recursion relation for members of $\{\mathfrak{I}_n\}$ that in order to maximize N^* we must have $N_{m-1} = N_{m-2} = 1$. Let k be a positive integer so that $m = 3k + j$, where $j = 1$, 2, or 3. The three pertinent identities in Lemma 1.1 imply that, for any of the three possible values of j, the maximal possible value of N^* is $\mathfrak{I}_m - 1$. Consequently, $N^* < \mathfrak{I}_m \leqslant M^*$, and it follows that $N < M$.

In [3], it was demonstrated that the positive integers can be represented in terms of the Tribonacci numbers by means of a "second canonical form," and it was proved that this new representation also gives rise to a lexicographic ordering. Analogous results hold for the sequence $\{\mathfrak{I}_n\}$. We begin by developing the second canonical form for a representation.

For each positive integer N, let $\mathfrak{I}_k$ be the least term of $\{\mathfrak{I}_n\}$ used in the Zeckendorf representation for N; of course, the subscript k depends on the particular integer N being examined. The uniqueness of the Zeckendorf representation implies it is possible to partition the positive integers into two sets as follows:

> S_1 is the set of all positive integers N such that
> $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$,

and

> S_2 is the set of all positive integers N such that
> $k \equiv 2 \pmod 3$.

Suppose the elements of the sets S_1 and S_2 are written in natural order, and let $S_{i,n}$ denote the n^{th} element in the set S_i for $i = 1$ or 2. We list the first ten entries in each set.

Table 1

n	$S_{1,n}$	$S_{2,n}$
1	$1 = \mathfrak{J}_1$	$2 = \mathfrak{J}_2$
2	$3 = \mathfrak{J}_3$	$5 = \mathfrak{J}_3 + \mathfrak{J}_2$
3	$4 = \mathfrak{J}_3 + \mathfrak{J}_1$	$8 = \mathfrak{J}_4 + \mathfrak{J}_2$
4	$6 = \mathfrak{J}_4$	$11 = \mathfrak{J}_5$
5	$7 = \mathfrak{J}_4 + \mathfrak{J}_1$	$13 = \mathfrak{J}_5 + \mathfrak{J}_2$
6	$9 = \mathfrak{J}_4 + \mathfrak{J}_3$	$16 = \mathfrak{J}_5 + \mathfrak{J}_3 + \mathfrak{J}_2$
7	$10 = \mathfrak{J}_4 + \mathfrak{J}_3 + \mathfrak{J}_1$	$19 = \mathfrak{J}_5 + \mathfrak{J}_4 + \mathfrak{J}_2$
8	$12 = \mathfrak{J}_5 + \mathfrak{J}_1$	$22 = \mathfrak{J}_6 + \mathfrak{J}_2$
9	$14 = \mathfrak{J}_5 + \mathfrak{J}_3$	$25 = \mathfrak{J}_6 + \mathfrak{J}_3 + \mathfrak{J}_2$
10	$15 = \mathfrak{J}_5 + \mathfrak{J}_3 + \mathfrak{J}_1$	$28 = \mathfrak{J}_6 + \mathfrak{J}_4 + \mathfrak{J}_2$

Theorem 2.2: The sets S_1 and S_2 can be characterized as follows:

> S_1 is the set of all positive integers N which can be represented in the form $\mathfrak{J}_1 + N_2\mathfrak{J}_2 + N_3\mathfrak{J}_3 + \ldots$, where each $N_i \in \{0, 1\}$ and $N_iN_{i+1}N_{i+2} = 0$ if $i > 1$;
>
> S_2 is the set of all positive integers N which can be represented in the form $\mathfrak{J}_2 + N_3\mathfrak{J}_3 + N_4\mathfrak{J}_4 + \cdots$, where each $N_i \in \{0, 1\}$ and $N_iN_{i+1}N_{i+2} = 0$ if $i > 2$.

Moreover, every positive integer has a unique representation in one of the above two forms.

Proof: Let N be a positive integer and let $\mathfrak{J}_k$ be the least member of $\{\mathfrak{J}_n\}$ used in the Zeckendorf representation of N in terms of $\{\mathfrak{J}_n\}$. There are three cases to consider depending on whether k is congruent to 0, 1, or 2 modulo 3.

If $k \equiv 0 \pmod 3$, then N is an element of S_1 and, for some nonnegative integer m, $k = 3m + 3$. Using the identities in Lemma 1.1 and the Zeckendorf representation for N, the term $\mathfrak{J}_k$ can be replaced by

$$(\mathfrak{J}_1 + \mathfrak{J}_2) + (\mathfrak{J}_4 + \mathfrak{J}_5) + \cdots + (\mathfrak{J}_{3m+1} + \mathfrak{J}_{3m+2});$$

moreover, this is the only admissible representation for $\mathfrak{J}_k$. These observations and the uniqueness of the Zeckendorf representation imply the uniqueness of this new representation for N.

If $k \equiv 1 \pmod 3$, again N lies in S_1 and, for some nonnegative integer m, $k = 3m + 1$. In this case, $\mathfrak{J}_k$ must be replaced by

$$\mathfrak{J}_1 + (\mathfrak{J}_2 + \mathfrak{J}_3) + (\mathfrak{J}_5 + \mathfrak{J}_6) + \cdots + (\mathfrak{J}_{3m-1} + \mathfrak{J}_{3m}).$$

This illustrates the reason for permitting $N_1N_2N_3 = 1$. Again, this new representation for N is the unique allowable representation.

Finally, if $k \equiv 2 \pmod 3$, then N lies in S_2 and, for some nonnegative integer m, $k = 3m + 2$. From Lemma 1.1, we have

$$\mathfrak{J}_k = 1 + \mathfrak{J}_1 + (\mathfrak{J}_3 + \mathfrak{J}_4) + (\mathfrak{J}_6 + \mathfrak{J}_7) + \cdots + (\mathfrak{J}_{3m} + \mathfrak{J}_{3m+1})$$

$$\mathfrak{J}_k = \mathfrak{J}_2 + (\mathfrak{J}_3 + \mathfrak{J}_4) + (\mathfrak{J}_6 + \mathfrak{J}_7) + \cdots + (\mathfrak{J}_{3m} + \mathfrak{J}_{3m+1}).$$

In this case, we see that $N_1N_2N_3 = 1$ may be necessary in representing some integers. The uniqueness of this new representation for N follows as in the previous cases.

The preceding theorem suggests a definition for a second canonical representation with respect to $\{\mathfrak{J}_n\}$: a positive integer N is being represented in *second canonical form* in terms of the sequence $\{\mathfrak{J}_n\}$ if, for some m,

$$N = N_1\mathfrak{J}_1 + N_2\mathfrak{J}_2 + N_3\mathfrak{J}_3 + \cdots + N_m\mathfrak{J}_m,$$

where (1) each $N_i \in \{0, 1\}$,
(2) at least one of N_1 and N_2 is nonzero,
(3) if $N_1 = 1$, then $N_i N_{i+1} N_{i+2} = 0$ for all $i > 1$,
and (4) if $N_2 = 1$, then $N_i N_{i+1} N_{i+2} = 0$ for all $i > 2$.

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3: Every positive integer can be uniquely represented in second canonical form in terms of the Hoggatt sequence of degree 3.

In [3], it is noted that the representation of the positive integers in second canonical form with respect to the Tribonacci numbers is a lexocigraphic ordering. Although the second canonical form of a representation with respect to $\{\mathfrak{J}_n\}$ is not defined in the same way as the second canonical form with respect to $\{T_n\}$, the two forms are similar and an analogous theorem holds for the third-degree Hoggatt numbers.

Theorem 2.4: The second canonical representation of the positive integers in terms of the sequence $\{\mathfrak{J}_n\}$ is a lexicographic ordering.

Proof: We begin as in the proof of Theorem 2.1.

Let M and N be two positive integers expressed in second canonical form in terms of $\{\mathfrak{J}_n\}$. There is some positive integer t such that, in second canonical form,

$$M = \sum_{i=1}^{t} M_i\mathfrak{J}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i\mathfrak{J}_i.$$

Let m be a positive integer such that $M_i = N_i$ for all $i > m$; further, suppose $M_m = 1$ and $N_m = 0$. Consider the following truncations of M and N:

$$M^* = M_1\mathfrak{J}_1 + M_2\mathfrak{J}_2 + \cdots + M_{m-1}\mathfrak{J}_{m-1} + \mathfrak{J}_m$$

and

$$N^* = N_1\mathfrak{J}_1 + N_2\mathfrak{J}_2 + \cdots + N_{m-1}\mathfrak{J}_{m-1}.$$

Since M has been represented in second canonical form, either M_1 or M_2 is nonzero; therefore, $M^* \geqslant \mathfrak{J}_1 + \mathfrak{J}_m > \mathfrak{J}_m$. Again, in order or maximize N^*, we must have $N_{m-1} = N_{m-2} = 1$. Let K be a positive integer such that $m = 3k + j$ for some $j = 1, 2$, or 3. Conisder the three appropriate identities in Lemma 1.1, and the three possible values of j.

If $m = 3k + 1$, then the maximum possible value of N^* is

$$\mathfrak{J}_{3k+1} - 1 + \mathfrak{J}_1 = \mathfrak{J}_{3k+1} = \mathfrak{J}_m.$$

If $m = 3k + 2$, then the maximum value for N^* is

$$\mathfrak{J}_{3k+2} - 1 = \mathfrak{J}_m - 1.$$

Finally, if $m = 3k + 3$, then the maximum possible N^* is

$$\mathfrak{J}_{3k+3} - 1 + \mathfrak{J}_1 = \mathfrak{J}_{3k+3} = \mathfrak{J}_m.$$

In any case, N^* does not exceed $\mathfrak{J}_m$ in value, and we have $N^* \leqslant \mathfrak{J}_m < M^*$; consequently, $N < M$.

Before proceeding to the generalizations of the preceding theorems in this section to degree r, we note a special property of the third-degree Hoggatt sequence.

Let $S_1, S_2, \ldots, S_n$ be nonempty sequences of positive integers such that every positive integer appears exactly once in exactly one of the sequences; in [1], such sequences are called *complementary* or a *complementary system*. In [3], properties of $\{T_n\}$ and a theorem of Lamdek and Moser [4] are used to demonstrate the existence of a pair of complementary sequences $\{X_n\}$ and $\{Y_n\}$ in natural order with the property that $\{X_n + Y_n\}$ and $\{Y_n - X_n\}$ is another pair of complementary sequences of positive integers in natural order. In the next theorem, we prove the existence and uniqueness of $\{X_n\}$ and $\{Y_n\}$.

Theorem 2.5: There exist exactly two sequences, $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$, of positive integers in natural order such that $\{X_n\}$ and $\{Y_n\}$ are complementary sequences and the sequences $\{X_n + Y_n\}$ and $\{Y_n - X_n\}$ are also complementary sequences in natural order.

Proof: We develop four sequences $\{X_n\}$, $\{Y_n\}$, $\{P_n\}$, and $\{Q_n\}$ as follows: let $X_1 = 1$, $P_1 = 1$, $Y_1 = X_1 + P_1 = 2$, and $Q_1 = X_1 + Y_1 = 3$; in general, to find X_n, P_n, Y_n, and Q_n, let

(1) X_n = the first positive integer not yet appearing as an X_i or a Y_i,
(2) P_n = the first positive integer not yet appearing as a P_i or a Q_i,
(3) $Y_n = X_n + P_n$, and
(4) $Q_n = X_n + Y_n$.

The following array arises.

Table 2

n	X_n	P_n	Y_n	Q_n
1	1	1	2	3
2	3	2	5	8
3	4	4	8	12
4	6	5	11	17
5	7	6	13	20
6	9	7	16	25
7	10	9	19	29
8	12	10	22	34
9	14	11	25	39
10	15	13	28	43
⋮	⋮	⋮	⋮	⋮

Note that (1)-(4) guarantee that $\{X_n\}$ and $\{Y_n\}$ are complementary sequences in natural order, as are $\{P_n\}$ and $\{Q_n\}$. From (3) and (4) it follows that

$$\{P_n\} = \{Y_n - X_n\} \quad \text{and} \quad \{Q_n\} = \{X_n + Y_n\},$$

as desired. Hence, the exixtence of the sequences $\{X_n\}$ and $\{Y_n\}$ has been established.

To verify the uniqueness of the sequences $\{X_n\}$ and $\{Y_n\}$, we note that the method of generating the four sequences yields exactly one pair of sequences satisfying the conditions in the statement of the theorem; therefore, any other pair of sequences satisfying these conditions must be obtained by some method other than that used to generate $\{X_n\}$ and $\{Y_n\}$.

Suppose there is another pair of sequences, denoted $\{\overline{X}_n\}$ and $\{\overline{Y}_n\}$, satisfying the conditions of the theorem. Let $\{\overline{Q}_n\}$ and $\{\overline{P}_n\}$ represent, respectively, the sum and difference sequences $\{\overline{X}_n + \overline{Y}_n\}$ and $\{\overline{Y}_n - \overline{X}_n\}$; it follows that properties (3) and (4) hold for the four new sequences. Suppose property (1) does not hold. Then, for some n, $\overline{X}_n$ is not the first positive integer not yet appearing as an $\overline{X}_i$ or a $\overline{Y}_i$; since $\{\overline{X}_n\}$ and $\{\overline{Y}_n\}$ are complementary and in natural order, $\overline{X}_n > \overline{Y}_n$. Consequently, $\overline{Y}_n - \overline{X}_n < 0$ and $\overline{P}_n$ is not a positive integer, a contradiction. Therefore, property (1) is necessary to the solution of the problem; similarly, property (2) must hold. Hence, the method used to generate $\{X_n\}$ and $\{Y_n\}$ provides the only pair of sequences satisfying the conditions of the theorem.

Consider the sets S_1 and S_2 defined earlier in this section. Recall that S_1 and S_2 are written in natural order, and $S_{i,n}$ denotes the n^{th} element of S_i for $i = 1$ or 2. We have seen that $\{S_{1,n}\}$ and $\{S_{2,n}\}$ are complementary sequences of positive integers in natural order. It has also been shown in [3] that

$$\{S_{1,n} + S_{2,n}\} \quad \text{and} \quad \{S_{2,n} - S_{1,n}\}$$

are complementary sequences in natural order. It follows that $\{S_{1,n}\}$ and $\{S_{2,n}\}$ are the sequences $\{X_n\}$ and $\{Y_n\}$ of Theorem 2.5. Therefore, the sets S_1 and S_2 can be generated by the method described in the proof of Theorem 2.5; no appeal to representations in terms of $\{\mathfrak{J}_n\}$ is necessary.

3. THE HOGGATT SEQUENCE OF DEGREE r

In this section, we note that the theorems of Section 2 involving lexicographic ordering have analogs for the r^{th}-degree Hoggatt sequence. Since the theorems of this section can be proved by using the same techniques as in Section 2, only sketches of proofs are given. Recall that from Section 1 we have r identities involving the sequence $\{\mathfrak{R}_n\}$ and a unique Zeckendorf representation for every positive integer in terms of $\{\mathfrak{R}_n\}$.

Theorem 3.1: The Zeckendorf representation of the positive integers in terms of the r^{th}-degree Hoggatt sequence $\{\mathfrak{R}_n\}$ is a lexicographic ordering.

Proof: Let M and N be two positive integers expressed in Zeckendorf form:

$$M = \sum_{i=1}^{t} M_i \mathfrak{R}_i \quad \text{and} \quad N = \sum_{i=1}^{t} N_i \mathfrak{R}_i,$$

where $M_i,\ N_i \in \{0,\ 1\}$,

$$M_1 M_2 \cdot \cdots \cdot M_{r-1} = N_1 N_2 \cdot \cdots \cdot N_{r-1} = 0,$$

and $$M_i M_{i+1} \cdot \cdots \cdot M_{i+r-1} = N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0 \text{ for all } i.$$

Let m be a positive integer such that $M_i = N_i$ for all $i > m$, let $M_m = 1$, and let $N_m = 0$. Consider the truncations M^* and N^* as in the proof of Theorem 2.1, and note that $M^* \geqslant \mathfrak{R}_m$. In order to maximize N^*, we must let

$$N_{m-1} = N_{m-2} = \cdots = N_{m-(r-1)} = 1.$$

From the r identities in Lemma 1.1, it follows that $N^* < \mathfrak{R}_m \leqslant M^*$, and consequently, $N < M$.

We next develop the second canonical form for a representation in terms of $\{\mathcal{R}_n\}$.

For a particular positive integer N, let $\mathcal{R}_k$ be the smallest term of $\{\mathcal{R}_n\}$ used in the Zeckendorf representation for N. Using the uniqueness of the Zeckendorf representation, the positive integers can be partitioned into $(r - 1)$ sets as follows:

> S_1 is the set of all positive integers N such that $k \equiv 0 \pmod r$ or $k \equiv 1 \pmod r$,

and for integers i such that $2 \leqslant i \leqslant r - 1$,

> S_i is the set of all positive integers N such that $k \equiv i \pmod r$.

Let the elements of the sets $S_1, S_2, \ldots, S_{r-1}$ be written in natural order.

Theorem 3.2: The sets $S_1, S_2, \ldots, S_{r-1}$ can be characterized as follows: for $j = 1, 2, \ldots, r - 1$,

> S_j is the set of all positive integers which can be represented in the form $N = \mathcal{R}_j + N_{j+1}\mathcal{R}_{j+1} + N_{j+2}\mathcal{R}_{j+2} + \cdots$, where each $N_i \in \{0, 1\}$ and $N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0$ if $i > j$.

Moreover, every positive integer has a unique representation in terms of $\{\mathcal{R}_n\}$ in one of these $(r - 1)$ forms.

Proof: Let N be a positive integer and let $\mathcal{R}_k$ be the least term of $\{\mathcal{R}_n\}$ used in the Zeckendorf representation of N. There are r cases to consider depending on whether k is congruent to $0, 1, 2, \ldots,$ or $(r - 1)$ modulo r. In each of these cases, the uniqueness of Zeckendorf representations and one of the identities in Lemma 1.1 yield the desired representation for N; moreover, the new representation is unique.

A positive integer N is represented in *second canonical form* in terms of the sequence $\{\mathcal{R}_n\}$ if, for some m,

$$N = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_m\mathcal{R}_m,$$

where

(1) each $N_i \in \{0, 1\}$,
(2) at least one of the coefficients $N_1, N_2, \ldots, N_{r-1}$ is nonzero, and
(3) if $N_j = 1$, then $N_i N_{i+1} \cdot \cdots \cdot N_{i+r-1} = 0$ for all $i > j$.

We immediately have the following corollary to Theorem 3.2.

Corollary 3.3: Every positive integer can be uniquely represented in second canonical form in terms of the sequence $\{\mathcal{R}_n\}$.

Finally, we have the analog to Theorem 2.4.

Theorem 3.4: The second canonical representation of the positive integers in terms of the sequence $\{\mathcal{R}_n\}$ is a lexicographic ordering.

Proof: With notation as in the proof of Theorem 3.1, but with the representation in second canonical form, consider the truncations of M and N:

$$M^* = M_1\mathcal{R}_1 + M_2\mathcal{R}_2 + \cdots + M_{m-1}\mathcal{R}_{m-1} + \mathcal{R}_m$$

and

$$N^* = N_1\mathcal{R}_1 + N_2\mathcal{R}_2 + \cdots + N_{m-1}\mathcal{R}_{m-1}.$$

One of the coefficients M_1, M_2, ..., M_{r-1} is nonzero; therefore,

$$M^* \geqslant \mathcal{R}_1 + \mathcal{R}_m > \mathcal{R}_m.$$

In order to maximize N^*, we let

$$N_{m-1} = N_{m-2} = \cdots = N_{m-(r-1)} = 1$$

and note that the identities in Lemma 1.1 imply that the maximum possible value for N^* is $\mathcal{R}_m$; therefore, $N^* \leqslant \mathcal{R}_m < M^*$ and $N < M$.

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