A CURIOUS SET OF NUMBERS

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1. INTRODUCTION

Recently, these two authors proved a theorem involving necessary and sufficient conditions on when a real ordinary differential expression can be made formally self-adjoint [1]. A differential expression

$$L(y) = \sum_{k=0}^{r} a_k(x)y^{(k)}(x)$$

is said to be *symmetric* or *formally self-adjoint* if $L(y) = L^+(y)$, where L^+ is the Lagrange adjoint of L defined by

$$L^+(y) = \sum_{k=0}^{r} (-1)^k (a_k(x)y(x))^{(k)}.$$

It is easy to see that if $L = L^+$ then it is necessary that r be even. If $L(y)$ is a differential expression and $f(x)$ is a function such that $f(x)L(y)$ is symmetric, then $f(x)$ is called a symmetry factor for $L(y)$. In [2], Littlejohn proved the following theorem.

Theorem: Suppose $a_k(x) \in C^k(I)$, $a_k(x)$ is real valued, $k = 0, 1, \ldots, 2n$, $a_{2n}(x) \neq 0$, where I is some interval of the real line. Then there exists a symmetry factor $f(x)$ for the expression

$$L(y) = \sum_{k=0}^{2n} a_k(x)y^{(k)}(x)$$

if and only if $f(x)$ simultaneously satisfies the n differential equations

$$\sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} a_{2s}^{(2s-2k+1-j)} f^{(j)} - a_{2k-1}f = 0, \tag{1}$$

$k = 1, 2, \ldots, n$, where B_{2i} is the Bernoulli number defined by

$$\frac{x}{e^x - 1} = 2 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2i}x^{2i}}{(2i)!}. \quad \blacksquare$$

However, these two authors have significantly improved the n equations that the symmetry factor must satisfy [1]. Directly from the definition of symmetry it is easy to see that $f(x)$ is a symmetry factor for (1) if and only if $A_{k+1} = 0$, $k = 0, 1, \ldots (2n - 1)$, where

$$A_{k+1} = \sum_{j=0}^{2n-k} (-1)^{k+j} \binom{k+j}{j} (f(x)a_{k+j}(x))^{(j)} - f(x)a_k(x).$$

Littlejohn & Krall show that $A_{k+1} = 0$, $k = 0, 1, \ldots (2n - 1)$ if and only if

$$C_{k+1} = \sum_{i=2k+1}^{2n} (-1)^i \binom{i - k - 1}{k} (a_i(x)f(x))^{(i-2k-1)} = 0, \tag{2}$$

$k = 0, 1, \ldots (n - 1)$. If we express the C_k's in terms of the A_k's, we see that:

$$C_1' = A_1,$$

$$C_2'' + 2C_1 = A_2,$$

and, for $3 \leqslant k \leqslant 2n - 1$,

$$1C_k + kC_{k-1}^{(k-2)} + \sum_{j=3}^{\left[\frac{k+3}{2}\right]} \frac{k(k-j)(k-j-1)(k-j-2)\ldots(k-2j+3)}{(j-1)!} C_{k-j+1}^{(k-2j+2)} = A_k,$$

where $C_k = 0$ if $k > n$ and $[\cdot]$ denotes the greatest integer function.

From the coefficients of these equations, we get the following array:

1st row	1	0						
2nd row	1	2						
3rd row	1	3	0					
4th row	1	4	2					
5th row	1	5	5	0				
6th row	1	6	9	2				
7th row	1	7	14	7	0			
8th row	1	8	20	16	2			
9th row	1	9	27	30	9	0		
10th row	1	10	35	50	25	2		
11th row	1	11	44	77	55	11	0	
12th row	1	12	54	112	105	36	2	
13th row	1	13	65	156	182	91	13	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

This array has many interesting properties, some of which we shall discuss in this note.

2. PROPERTIES OF THE ARRAY

If we add all of the entries in each row, we arrive at the sequence

1, 3, 4, 7, 11, 18, 29, 47, 76, ...

A Fibonacci sequence! (Actually, this sequence is called the Lucas sequence.) From this, we can desily derive

Theorem 1: For $n \geqslant 3$,

$$1 + n + \sum_{j=3}^{\left[\frac{n+3}{2}\right]} \frac{n(n-j)(n-j-1)(n-j-2)\ldots(n-2j+3)}{(j-1)!}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \blacksquare$$

For $n \geqslant 3$ and $j \geqslant 3$, the number

$$A_{n,j} = \frac{n(n-j)(n-j-1)\ \ldots\ (n-2j+3)}{(j-1)!}$$

is the entry in the n^{th} row and j^{th} column. Alternatively, $A_{n,j}$ is the r^{th} element in the j^{th} column where $r = n - 2j + 4$. We now show how to obtain any element in the j^{th} column by looking at the $(j - 1)^{\text{st}}$ column. Consider, for example, $A_{11,4} = 77$, which is the *seventh entry* in the fourth column. Observe that we can also obtain 77 by adding the first *seven* entries in the third column:

$$77 = 0 + 2 + 5 + 9 + 14 + 20 + 27.$$

As another example, $A_{13,6} = 91$, the *fifth* number in the sixth column can also be obtained by adding the first *five* numbers in the fifth column:

$$91 = 0 + 2 + 9 + 25 + 55.$$

From this, we get

Theorem 2: For $n \geqslant 4$ and $3 \leqslant j \leqslant \left[\frac{n+3}{2}\right]$,

$$\sum_{i=2j-4}^{n-2} i(i-j+1)(i-j)\ \ldots\ (i-2j+5) = \frac{n(n-j)(n-j-1)\ \ldots\ (n-2j+3)}{j-1}. \blacksquare$$

Of course, this process can also be reversed; that is, we can obtain the entries in the $(j - 1)^{\text{st}}$ column by looking at the j^{th} column. More specifically, by taking differences of successive elements in the j^{th} column, we obtain the entries in the $(j - 1)^{\text{st}}$ column. The reason for this is the identity

$$A_{n,j} - A_{n-1,j} = A_{n-2,j-1}.$$

There are probably many other patterns appearing in this array; we list a few more:

$$1 + n + A_{n+1,3} + A_{n+2,4} + A_{n+3,5} + \cdots + A_{2n-1,n+1}$$
$$= 3 \cdot 2^{n-2},\ n \geqslant 2, \tag{3}$$
$$A_{2n,n} = n^2. \tag{4}$$

How many new patterns can you find?

The first set of necessary and sufficient conditions for the existence of a symmetry factor [i.e., equation (1)] involve the Bernoulli numbers. We have shown that the second set of conditions [equation (2)], which are equivalent to the first set, involve the Fibonacci numbers. What is the connection between these two sets of numbers?

ACKNOWLEDGMENTS

The authors wish to thank the referee for many helpful suggestions as well as the referee's student who found a few more patterns in the array. If we add the entries on the (main) diagonals, we obtain the sequence

$$1, 1, 3, 4, 5, 8, 12, 17, 25, 37, 54, 79, \ldots$$

or $a_n = a_{n-1} + a_{n-3}$, $n \geqslant 4$. Another interesting pattern is the following:

$$A_{n,j} = \sum_{k=1}^{j} A_{n-2j+2k-1,k}, \tag{5}$$

where we adopt the notation that $A_{0,1} = 1$, $A_{n,k} = 0$ when $n < 0$ and $A_{n,k} = 0$ if

$$k > \left[\frac{n+2}{2}\right].$$

For example,

$$A_{11,5} = A_{2,1} + A_{4,2} + A_{6,3} + A_{8,4} + A_{10,5} = 1 + 4 + 9 + 16 + 25 = 55.$$

REFERENCES

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