THE K^{th}-ORDER ANALOG OF A RESULT OF L. CARLITZ

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This note is an extension of the work of Carlitz [1] and of Laohakosol and Roenrom [2]. The proofs given here are very similar to those of Laohakosol and Roenrom as presented in [2].

Consider the k^{th}-order difference equation

$$\sum_{j=0}^{k}\sum_{m=0}^{j}(-1)^{m+k-j}\binom{j}{m}p^m n^{(m)}a_{k-j}f_{n+j-m}(x) = x^{k-1}f_{n+k-1}(x) \tag{1}$$

for all $n = 0, 1, 2, \ldots$, with initial conditions

$$f_0(x) = f_1(x) = \cdots = f_{k-2}(x) = 0,\ f_{k-1}(x) = 1, \tag{2}$$

and $a_0 = 1$; a_i $(i = 1, 2, \ldots, k)$ are arbitrary parameters, where

$$n^{(m)} = n(n-1)\ldots(n-m+1)$$

subject to the following three restrictions:

I. $p \neq 0$.

II. All k roots α_i $(i = 1, 2, \ldots, k)$ of the equation $G(0, \alpha, k) = 0$ are distinct and none is a nonpositive integer, where

$$G(r, \alpha, k) = \sum_{j=0}^{k}(-1)^{k-j}a_{k-j}p^j(\alpha + r + j - 1)^{(j)}.$$

III. All $k - 1$ roots r_i $(i = 1, 2, \ldots, k - 1)$ of the equation $L(r, \alpha, k) = 0$ are nonpositive integers, where α denotes any one of $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ or α_k from II and

$$L(r, \alpha, k) = \{G(r, \alpha, k) - G(0, \alpha, k)\}/r.$$

Let

$$F(t) := F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n/(n!) \tag{3}$$

be the exponential generating function for $f_n(x)$. From (1)-(3), we have

$$\sum_{j=0}^{k}(-1)^{k-j}(1 - pt)^j a_{k-j}F^{(j)}(t) = x^{k-1}F^{(k-1)}(t).$$

Next, we define an operator

$$\Delta := \sum_{j=0}^{k}(-1)^{k-j}(1 - pt)^j a_{k-j}D^j \quad \text{(where } D = d/dt\text{)}.$$

Then our differential equation becomes

$$\Delta F(t) = x^{k-1}F^{(k-1)}(t).$$

We expect k independent solutions of this differential equation to be of the form

$$\phi(t, \alpha) := \phi(t, \alpha, x) = \sum_{m=0}^{\infty} T_m x^m (1 - pt)^{-\alpha - m},$$

where α is any one of $\alpha_1, \alpha_2, \ldots, \alpha_k$. Thus, we must compute $T_m = T_m(\alpha)$.

Using a method similar to that given in [2], we derive

$$T_{j(k-1)+i} = \frac{(\alpha + jk - j + i - 1)^{j(k-1)}}{p^j \left\{ \prod_{m=1}^{j} (mk - m + i) \left[\prod_{s=1}^{k-1} (mk - m + i - r) \right] \right\}} T_i$$

for all $i = 0, 1, \ldots, k - 2$.

Let $C_n(\alpha) := C_n(x, \alpha)$ be the coefficient of $t^n/(n!)$ in $\phi(t, \alpha)$, then

$$C_n(\alpha) = \sum_{j=0}^{\infty} T_{j(k-1)}(\alpha + jk - j + n - 1)^{(n)} p^n x^{j(k-1)}.$$

Hence, we have the general solution of (1) as

$$f_n(x) = \sum_{i=0}^{k} w_i C_n(x, \alpha_i)$$

where

$$w_i = w_i(\alpha_1, \alpha_2, \ldots, \alpha_k) \qquad (i = 1, 2, \ldots, k)$$

are to be chosen so that the initial conditions (2) are fulfilled, namely

$$C \cdot W = E$$

where

$$C = \begin{bmatrix} C_0(\alpha_1) & C_0(\alpha_2) & \cdots & C_0(\alpha_k) \\ C_1(\alpha_1) & C_1(\alpha_2) & \cdots & C_1(\alpha_k) \\ \vdots & \vdots & & \vdots \\ C_{k-1}(\alpha_1) & C_{k-1}(\alpha_2) & \cdots & C_{k-1}(\alpha_k) \end{bmatrix}_{k*k}, \quad W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}_{k*1}, \quad E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{k*1},$$

and $\det C \neq 0$. Using Cramer's rule, we obtain the solution of W. With these values, we have completely solved (1). Obviously, the difference equations of [1] and [2] are the special cases of (1).

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