ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-628 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

What is the present average age of Fibonacci's rabbits? (Recall that he introduced a pair of mature rabbits at the beginning of his year and that rabbits mature in their second month. Further, no rabbits died. Let us say that he did this at the beginning of 1202 and that he introduced a pair of one-month-old rabbits. At the end of the first month, this pair would have matured and produced a new pair, giving us a pair of 2-month-old rabbits and a pair of 0-month-old rabbits. At the end of the second month we have a pair of 3-month-old rabbits and pairs of 1-month-old and of 0-month-old rabbits.) Before solving the problem, make a guess at the answer.

B-629 *Proposed by Mohammad K, Azarian, Univ. of Evansville, Evansville, IN*

For which integers a, b, and c is it possible to find integers x and y satisfying

$$(x + y)^2 - cx^2 + 2(b - a + ac)x - 2(a - b)y + (a - b)^2 - ca^2 = 0?$$

B-630 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a and b be constants and define sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ by $A_1 = a$, $A_2 = b$, $B_1 = 2b - a$, $B_2 = 2a + b$, and $A_n = A_{n-1} + A_{n-2}$ and $B_n = B_{n-1} + B_{n-2}$ for $n \geqq 3$.

(i) Determine a and b so that $(A_n + B_n)/2 = [(1 + \sqrt{5})/2]^n$.

(ii) For these a and b, obtain $(B_n + A_n)/(B_n - A_n)$.

B-631 *Proposed by L. Kuipers, Sierre, Switzerland*

For N in $\{1, 2, \ldots\}$ and $N \geqq m + 1$, obtain, in closed form,

$$u_N = \sum_{k=m+1}^{m+N} k(k-1) \cdots (k-m)\binom{n+k}{k}.$$

B-632 *Proposed by H.-J. Seiffert, Berlin, Germany*

Find the determinant of the n by n matrix (x_{ij}) with $x_{ij} = (1 + \sqrt{5})/2$ for $j > i$, $x_{ij} = (1 - \sqrt{5})/2$ for $j < i$, and $x_{ij} = 1$ for $j = i$.

B-633 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n \geqq 2$ be an integer and define

$$A_n = \sum_{k=0}^{\infty} \frac{F_k}{n^k}, \quad B_n = \sum_{k=0}^{\infty} \frac{L_k}{n^k}.$$

Prove that $B_n/A_n = 2n - 1$.

SOLUTIONS

Recurrence Relation for Squares

B-604 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let c be a fixed number and $u_{n+2} = cu_{n+1} + u_n$ for n in $N = \{0, 1, 2, \ldots\}$. Show that there exists a number h such that

$$u_{n+4}^2 = hu_{n+3}^2 - hu_{n+1}^2 + u_n^2 \text{ for } n \text{ in } N.$$

Solution by Demetris Antzoulakos, Univ. of Patras, Patras, Greece

We shall show that $h = c^2 + 2$.

Using successively the above recurrence relation, we get:

$$\begin{aligned}
u_{n+4}^2 &= c^2u_{n+3}^2 + u_{n+2}^2 + 2cu_{n+3}u_{n+2} = c^2u_{n+3}^2 + u_{n+2}^2 + 2u_{n+3}^2 - 2u_{n+3}u_{n+1} \\
&= (c^2 + 2)u_{n+3}^2 + u_{n+2}^2 - 2cu_{n+1}u_{n+2} - 2u_{n+1}^2 \\
&= (c^2 + 2)u_{n+3}^2 + c^2u_{n+1}^2 + u_n^2 + 2cu_{n+1}u_n - 2c^2u_{n+1}^2 - 2cu_{n+1}u_n - 2u_{n+1}^2 \\
&= (c^2 + 2)u_{n+3}^2 - (c^2 + 2)u_{n+1}^2 + u_n^2.
\end{aligned}$$

Note: The above recurrence includes the exponent 2 dropped by the E.P.S. editor.

Also solved by Paul S. Bruckman, Piero Filipponi, Herta T. Freitag, C. Georghiou, L. Kuipers, Sahib Singh, and the proposer.

Never Prime

B-605 *Proposed by Herta T. Freitag, Roanoke, VA*

Let
$$S(n) = \sum_{i=1}^{n} L_{2n+2i-1}.$$

Determine the positive integers n, if any, for which $S(n)$ is prime.

Solution by Paul S. Bruckman, Fair Oaks, CA

First, we obtain a closed form for $S(n)$. Since

$$S(n) = \sum_{i=1}^{n} (L_{2n+2i} - L_{2n+2i-2}),$$

thus,

$$S(n) = L_{4n} - L_{2n}. \tag{1}$$

Also, $L_{4n} = L_{2n}^2 - 2$. Hence,

$$S(n) = L_{2n}^2 - L_{2n} - 2. \tag{2}$$

In turn, this implies

$$S(n) = (L_{2n} - 2)(L_{2n} + 1).$$

Note that $S(1) = (3 - 2)(3 + 1) = 4$, which is not prime; also, each factor of $S(n)$ is greater than 1 if $n > 1$. Therefore, $S(n)$ is composite for all n.

Also solved by Frank Cunliffe, Piero Filipponi, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Very Much Simplified

B-606 *Proposed by L. Kuipers, Sierre, Switzerland*

Simplify the expression
$$L_{n+1}^2 + 2L_{n-1}L_{n+1} - 25F_n^2 + L_{n-1}^2.$$

Solution by Gregory Wulczyn, Lewisburg, PA

$$\begin{aligned} L_{n+1}^2 + 2L_{n-1}L_{n+1} + L_{n-1}^2 - 25F_n^2 &= (L_{n+1} + L_{n-1})^2 - 25F_n^2 \\ &= (5F_1F_n)^2 - 25F_n^2 = 0. \end{aligned}$$

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Frank Cunliffe, Piero Filipponi, Herta T. Freitag, C. Georghiou, Hans Kappus, Joseph J. Kostal, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Product of Exponential Generating Functions

B-607 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let
$$C_n = \sum_{k=0}^{n} \binom{n}{k} F_k L_{n-k}.$$

Show that $C_n/2^n$ is an integer for n in $\{0, 1, 2, \ldots\}$.

Solution by Bob Prielipp, Univ. of Wisconsin-Oshkosh, WI

Since $F_k = (\alpha^k - \beta^k)/\sqrt{5}$ and $L_{n-k} = \alpha^{n-k} + \beta^{n-k}$ where $\alpha = (1 + \sqrt{5})/2)$ and $\beta = (1 - \sqrt{5})/2$, $F_k L_{n-k} = (\alpha^n - \beta^n)/\sqrt{5} - (\alpha^{n-k}\beta^k)/\sqrt{5} + (\beta^{n-k}\alpha^k)/\sqrt{5}$. Hence,

$$C_n = \sum_{k=0}^{n} \binom{n}{k} F_n - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k}\beta^k + \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \beta^{n-k}\alpha^k$$

$$= 2^n F_n - \frac{1}{\sqrt{5}}(\alpha + \beta)^n + \frac{1}{\sqrt{5}}(\beta + \alpha)^n$$

$\left[\text{using the fact that } \sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ and the Binomial Theorem}\right]$

$$= 2^n F_n.$$

The required result follows.

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Frank Cunliffe, Russell Euler, Piero Filipponi, Herta T. Freitag, C. Georghiou, Hans Kappus, Joseph J. Kostal, L. Kuipers, H.-J. Seiffert, Sahib Singh, Gregory Wulczyn, and the proposer.

Integral Average of Squares

B-608 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

For $k = \{2, 3, \ldots\}$ and n in $N = \{0, 1, 2, \ldots\}$, let

$$S_{n,k} = \frac{1}{k} \sum_{j=n}^{n+k-1} F_j^2$$

denote the quadratic mean taken over k consecutive Fibonacci numbers of which the first is F_n. Find the smallest such $k \geqq 2$ for which $S_{n,k}$ is an integer for all n in N.

Solution by Philip L. Mana, Albuquerque, NM

Since $S_{1,k} - S_{0,k} = F_k^2/k$, a necessary condition on k is that $k | F_k^2$. The two smallest such k in $\{2, 3, \ldots\}$ are 5 and 12. $S_{0,5}$ and $S_{1,5}$ are integers but $S_{2,5}$ is not since $F_6^2 \not\equiv F_1^2 \pmod 5$. Thus, 5 is not a solution.

It is known that

$$\sum_{j=0}^{m-1} F_j^2 = F_m F_{m-1}.$$

Hence,

$$S_{nk} = (F_{n+k}F_{n+k-1} - F_k F_{k-1})/k.$$

Since $F_{12} = 144 \equiv 0 \pmod{12}$ and $F_{13} = 233 \equiv 5 \pmod{12}$, it follows by induction that $F_{n+12} \equiv 5F_n \pmod{12}$. This implies that $F_{n+12}F_{n+11} \equiv 25F_n F_{n-1} \pmod{12}$ and hence $S_{n,12}$ is an integer for all n in N. Thus, $k = 12$ is a solution.

Note: P. S. Bruckman points out that $S_{n,k}$ is a "mean of squares" rather than a "quadratic mean."

Also solved by Paul S. Bruckman, Frank Cunliffe, Herta T. Freitag, C. Georghiou, L. Kuipers, Chris Long, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

Sum of Squares

B-609 *Proposed by Adina DiPorto & Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Find a closed form expression for

$$S = \sum_{k=1}^{n} (kF_k)^2$$

and show that $S_n \equiv n(-1)^n \pmod{F_n}$.

Solution by C. Georghiou, Univ. of Patras, Patras, Greece

We will show that $S_n \equiv n(-1)^{n+1} \pmod{F_n}$.

Let $f(x) = x + x^2 + \cdots + x^n$ and $g(x) = 1^2x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n$. We then have $g(x) = x^2f''(x) + xf'(x)$ and, therefore,

$$S_n = (g(\alpha^2) + g(\beta^2) - 2g(-1))/5$$

$$= \frac{1}{5}[(n-1)^2L_{2n+1} + (2n-1)L_{2n-1} - n(n+1)(-1)^n]$$

and by using the identity

$$L_{2n-1} = 5F_nF_{n-1} - (-1)^n,$$

we get

$$S_n = (n-1)^2F_n^2 + (n^2+2)F_nF_{n-1} - n(-1)^n,$$

from which the assertion follows.

Note: The solver corrected back to the proposer's $S_n \equiv n(-1)^{n+1}$.

Also solved by Paul S. Bruckman, Herta T. Freitag, Hans Kappus, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

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