A NOTE ON SPECIALLY MULTIPLICATIVE ARITHMETIC FUNCTIONS

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An arithmetic function f is called multiplicative if

$$f(mn) = f(m)f(n), \tag{1}$$

whenever $(m, n) = 1$. A multiplicative function f is called completely multiplicative if (1) holds for all m, n. Further, a multiplicative function f is said to be a quadratic (see [1], [3], [8]) or a specially multiplicative function (see [2], [4], [6], [7]) if

$$f = a \circ b, \tag{2}$$

where a, b are completely multiplicative functions and $\circ$ denotes the Dirichlet product. It is known that (2) is equivalent to

$$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)g(d)\mu(d),$$

where g is a completely multiplicative function and μ denotes the Möbius function. The completely multiplicative function g is defined for every prime by

$$g(p) = (ab)(p) \quad \text{or} \quad g(p) = f(p)^2 - f(p^2) \quad \text{or} \quad g(p) = f^{-1}(p^2),$$

where f^{-1} denotes the Dirichlet inverse of f. Since a quadratic f is multiplicative, the values $f(n)$ are known if the values $f(p^m)$ are known for all primes p and all positive integers m. Furthermore, the values $f(p^m)$ are known if the values $f(p)$, $f(p^2)$ [or the values $f(p)$, $f^{-1}(p^2)$ or the values $a(p)$, $b(p)$] are known. The values $f(p^m)$ are given recursively by

$$f(1) = 1,$$
$$f(p),\ f(p^2) \text{ are arbitrary},$$
$$f(p^m) = f(p)f(p^{m-1}) - g(p)f(p^{m-2}),\ m = 3, 4, \ldots . \tag{3}$$

Consequently, if we put $f(p^m) = S_m$, we obtain a generalized Fibonacci sequence determined by

$$S_0 = 1,$$
$$S_1,\ S_2 \text{ are arbitrary},$$
$$S_{m+1} = S_1S_m - ((S_1)^2 - S_2)S_{m-1},\ m = 2, 3, 4, \ldots .$$

If we let $S_1 = 1$, $S_2 = 2$, we obtain the Fibonacci sequence.

If f is specially multiplicative and $f = a \circ b$, where a, b are completely multiplicative, then the generating series of f to the base p is given by

$$f_{(p)}(x) = \frac{1}{(1 - \alpha x)(1 - \beta x)} \quad (p \text{ a prime}),$$

where $\alpha = a(p)$, $\beta = b(p)$. Then

$$f_{(p)}(x) = \frac{1}{1 - f(p)x + g(p)x^2},$$

where $f(p) = \alpha + \beta$ and $g(p) = \alpha\beta$. Noting that the generating function of the Fibonacci sequence $\{F_n\}$ is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2},$$

$f_{(p)}(x)$ will generate $\{F_n\}$ if $f(p) = 1$ and $g(p) = -1$.

If a is any nonzero complex number, one could consider f for which $f(p) = a$ and $g(p) = -a^2$. It will follow that

$$f_{(p)}(x) = \frac{1}{1 - ax - a^2x^2} = \sum_{n=0}^{\infty} a^n F_n x^n.$$

Hence, $f(p^n) = a^n F_n$. Write $f(p^n) = G_n$. Using known properties (see [5], [9]) of the Fibonacci sequence $\{F_n\}$, for example, the following properties of the sequence $\{G_n\}$ can be derived:

$$\sum_{k=0}^{n} a^{n-k+2} G_k = G_{n+2} - a^{n+2},$$

$$\sum_{k=0}^{n} (-1)^k a^{n-k} G_k = (-1)^n a G_{n-1} + a^n,$$

$$\sum_{k=0}^{n} a^{2(n-k)+1} G_{2k} = G_{2n+1},$$

$$\sum_{k=1}^{n} a^{2(n-k)+1} G_{2k-1} = G_{2n} - a^{2n},$$

$$2\sum_{k=1}^{n} a^{3(n-k)+2} G_{3k-1} = G_{3n+1} - a^{3n+1},$$

$$\sum_{k=0}^{n} (n - k) a^{n-k+3} G_k = G_{n+3} - (n + 3)a^{n+3},$$

$$\sum_{k=0}^{2n} a^{2(2n-k)+1} G_k G_{k+1} = G_{2n+1}^2,$$

$$\sum_{k=0}^{2n-1} a^{2(2n-k)-1} G_k G_{k+1} = G_{2n}^2 - a^{4n},$$

$$\sum_{k=0}^{n} a^{2(n-k)+1} G_k^2 = G_n G_{n+1},$$

$$10\sum_{k=0}^{n} a^{3(n-k)+4}G_k^3 = G_{3n+4} + (-1)^n 6a^{2n+5}G_{n-1} + 5a^{3n+4},$$

$$G_{n+m} = G_n G_m + a^2 G_{n-1}G_{m-1},$$

$$G_n^2 - G_{n-k}G_{n+k} = (-1)^{n-k+1}G_{k-1}^2 a^{2(n-k+1)},$$

$$aG_{3n+2} = G_{n+1}^3 + a^3G_n^3 - a^6G_{n-1}^3,$$

$$aG_{2n+1} = G_{n+1}^2 - a^4G_{n-1}^2.$$

The proofs of the above relations are omitted.

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