A NOTE ON THE THIRD-ORDER STRONG DIVISIBILITY SEQUENCES

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A k^{th}-order linear recurrent sequence $\mathbf{u} = \{u_n : n = 1, 2, \ldots\}$ of integers satisfying the following equation for greatest common divisors,

$$(u_i, u_j) = |u_{(i,j)}| \quad \text{for all } i, j \geqslant 1, \tag{1}$$

is called a k^{th}-order strong divisibility sequence. A complete characterization of all the second-order strong divisibility sequences was given in [1] for integers and then in [3] for an arbitrary algebraic number field. In this note we shall study the third-order strong divisibility sequences.

The system of all the sequences of integers $\mathbf{u} = \{u_n : n = 1, 2, \ldots\}$ defined by

$$u_1 = 1, \quad u_2 = \nu, \quad u_3 = \mu, \tag{2}$$

$$u_{n+3} = a \cdot u_{n+2} + b \cdot u_{n+1} + c \cdot u_n \text{ for } n \geqslant 1 \tag{3}$$

(where ν, μ, a, b, c are integers) will be denoted by U. The system of all the strong divisibility sequences from U [i.e., sequences from U satisfying (1)] will be denoted by D.

The aim of this paper is to find all the strong divisibility sequences in certain subsystems of U and, further, to give some necessary conditions for a sequence from U to be a strong divisibility sequence. Notice that we may take $u_1 = 1$ without loss of generality because all the third-order strong divisibility sequences are obviously all the integral multiples of sequences from D.

1. THE CASES $u_2 = 0$ AND $u_3 = 0$

Let U_1 denote the system of all the sequences from U satisfying $u_2 = 0$ and let U_2 denote all the sequences from U satisfying $u_3 = 0$. Further, let

$$A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} \quad \text{and} \quad B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6\},$$

where

$$\mathbf{a}_1 = \{1, 0, 1, 0, 1, \ldots\} \qquad \mathbf{a}_2 = \{1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$$
$$\mathbf{a}_3 = \{1, 0, -1, 0, -1, \ldots\} \qquad \mathbf{a}_4 = \{1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots\}$$

$\mathbf{b}_1 = \{1, 1, 0, 1, 1, 0, \ldots\}$
$\mathbf{b}_2 = \{1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots\}$
$\mathbf{b}_3 = \{1, 1, 0, -1, 1, 0, -1, 1, 0, \ldots\}$
$\mathbf{b}_4 = \{1, -1, 0, -1, 1, 0, 1, -1, 0, \ldots\}$
$\mathbf{b}_5 = \{1, -1, 0, 1, -1, 0, \ldots\}$
$\mathbf{b}_6 = \{1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, \ldots\}$

Directly from the definitions, we get: $A \subseteq D \cap U_1$; $B \subseteq D \cap U_2$. The following propositions show that both the inclusions are, in fact, equalities, i.e., the sequences from A (from B) are precisely all the strong divisibility sequences from U_1 (from U_2).

Proposition 1.1: Let $\mathbf{u} = \{u_n\} \in U_1$. Then $\mathbf{u} \in D$ if and only if $\mathbf{u} \in A$.

Proof: Let $\mathbf{u} \in D$; then, from $(u_2, u_{2k}) = 0$ and $(u_2, u_{2k+1}) = 1$, we get $u_{2k} = 0$ and $u_{2k+1} = \pm 1$ for every $k \geqslant 1$. Now, from $u_3 = \pm 1$, $u_4 = 0$, $u_5 = \pm 1$, we obtain four cases:

(i) $u_3 = u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{a}_1$;
(ii) $u_3 = 1,\ u_5 = -1 \Rightarrow \mathbf{u} = \mathbf{a}_2$;
(iii) $u_3 = -1,\ u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{a}_4$;
(iv) $u_3 = u_5 = -1 \Rightarrow \mathbf{u} = \mathbf{a}_3$;

hence, we get $\mathbf{u} \in A$. The converse is obvious.

Proposition 1.2: Let $\mathbf{u} = \{u_n\} \in U_2$. Then $\mathbf{u} \in D$ if and only if $\mathbf{u} \in B$.

Proof: Let $\mathbf{u} \in D$; then, from

$$|u_n| = (u_3, u_n) = \begin{cases} |u_3| & \text{for } 3|n \\ |u_1| & \text{for } 3\nmid n \end{cases}, \text{ we get } u_n = \begin{cases} 0 & \text{for } 3|n \\ \pm 1 & \text{for } 3\nmid n \end{cases}.$$

Thus, $u_2 = \pm 1$, $u_4 = \pm 1$, $u_5 = \pm 1$, $u_6 = 0$, and we obtain eight cases:

(i) $u_2 = u_4 = u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{b}_1$;
(ii) $u_2 = u_4 = 1,\ u_5 = -1 \Rightarrow u_6 = 2$, a contradiction;
(iii) $u_2 = 1,\ u_4 = -1,\ u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{b}_3$;
(iv) $u_2 = 1,\ u_4 = u_5 = -1 \Rightarrow \mathbf{u} = \mathbf{b}_2$;
(v) $u_2 = -1,\ u_4 = u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{b}_6$;
(vi) $u_2 = -1,\ u_4 = 1,\ u_5 = -1 \Rightarrow \mathbf{u} = \mathbf{b}_5$;
(vii) $u_2 = u_4 = -1,\ u_5 = 1 \Rightarrow \mathbf{u} = \mathbf{b}_4$;
(viii) $u_2 = u_4 = u_5 = -1 \Rightarrow u_6 = -2$, a contradiction;

hence, we get $\mathbf{u} \in B$. Again, the converse is obvious.

2. THE CASE $u_2 \neq 0$, $u_3 \neq 0$

Let U_3 denote the system of all the sequences from U satisfying $u_2 \neq 0$ and $u_3 \neq 0$. Obviously: $U = U_1 \cup U_2 \cup U_3$ and $U_1 \cap U_3 = U_2 \cap U_3 = \emptyset$. Moreover, it is obvious that, for all the sequences from U, it holds that

$$(u_1, u_n) = |u_{(1,n)}| \quad \text{for all } n \geqslant 1.$$

Proposition 2.1: Let $\mathbf{u} = \{u_n\} \in U_3$. Then $(u_i, u_j) = |u_{(i,j)}|$ for $1 \leqslant i, j \leqslant 4$ if and only if the following conditions hold:

$$(\nu, \mu) = 1; \tag{4}$$

$$c = f \cdot \nu - a \cdot \mu, \text{ where } f \text{ is a fixed integer;} \tag{5}$$

$$(\mu, b + f) = 1. \tag{6}$$

Proof: Obviously $(u_2, u_3) = |u_1| \Leftrightarrow (\nu, \mu) = 1$ and $(u_2, u_4) = |u_2| \Leftrightarrow$ there exists an integer f such that $f\nu = a\mu + c$. Finally, let (4) and (5) hold; then,

$$(u_3, u_4) = |u_1| \Leftrightarrow (\mu, b\nu + f\nu) = 1 \Leftrightarrow (\mu, b + f) = 1.$$

Proposition 2.2: Let $\mathbf{u} = \{u_n\} \in U_3$. Then $(u_i, u_j) = |u_{(i,j)}|$ for $1 \leqslant i, j \leqslant 5$ if and only if (4), (5), (6), and the following conditions hold:

$$(\nu, b) = 1; \tag{7}$$

$$(\mu, \nu f + a \cdot (b + f)) = 1; \tag{8}$$

$$(b + f, \nu \cdot (\nu f - \mu a) + \mu b) = 1. \tag{9}$$

Proof: Let (4) and (5) hold; then,

$$u_4 = \nu \cdot (b + f), \; u_5 = a\nu(b + f) + b\mu + (f\nu - a\mu)\nu.$$

Thus, $u_5 \equiv b\mu \pmod{|\nu|}$ and we get $(u_2, u_5) = |u_1| \Leftrightarrow (\nu, b) = 1$. Furthermore, $u_5 \equiv \nu \cdot (ab + af + f\nu) \pmod{|\mu|}$ and, therefore,

$$(u_3, u_5) = |u_1| \Leftrightarrow (\mu, ab + af + f\nu) = 1.$$

Finally, let (4), (5), and (7) hold; then,

$$(u_4, u_5) = |u_1| \Leftrightarrow (\nu(b + f), \nu(\nu f - a\mu) + \mu b) = 1$$
$$\Leftrightarrow (b + f, \nu(\nu f - a\mu) + \mu b) = 1,$$

which completes the proof.

Proposition 2.3: Let $\mathbf{u} = \{u_n\} \in U_3$. Then $(u_i, u_j) = |u_{(i,j)}|$ for $1 \leqslant i, j \leqslant 6$ if and only if (4)-(9) and the following conditions hold:

$$\nu | a(b - \mu); \tag{10}$$

$$\mu \mid (\nu af + (a^2 + b)(b + f)); \tag{11}$$

$$\left(b + f,\ \nu af + \mu\left(f - a^2 + \frac{a(b - \mu)}{\nu}\right)\right) = 1; \tag{12}$$

$$(\nu(a(b + f - \mu) + f\nu) + \mu b,\ \nu((b + f)(a^2 + b) + a(f\nu - a\mu) + f\mu) + \mu a(b - \mu)) = 1. \tag{13}$$

Proof: Let (5) hold, then $u_4 = \nu \cdot (b + f)$; $u_5 = \nu \cdot (a(b + f - \mu) + f\nu) + \mu b$; $u_6 = \nu((b + f)(a^2 + b) + a(f\nu - a\mu) + f\mu) + \mu a(b - \mu)$; and obviously $(u_5, u_6) = |u_1| \Longleftrightarrow (13)$. Further, let (4) and (5) hold; then,

$$(u_2, u_6) = |u_2| \Longleftrightarrow (10) \quad \text{and} \quad (u_3, u_6) = |u_3| \Longleftrightarrow (11).$$

Finally, let (5) and (10) hold; then

$$(u_4, u_6) = |u_2| \Longleftrightarrow (12),$$

which completes the proof.

Lemma 2.4: Let $\mathbf{u} = \{u_n\} \in U_3$, $\mathbf{u}$ satisfying (5) and (10). Then

$$u_{2k} \equiv 0 \pmod{|\nu|}; \quad u_{2k+1} \equiv b^{k-1} \cdot \mu \pmod{|\nu|} \quad \text{for all } k \geqslant 1. \tag{14}$$

Proof: From (5) and (10), we get: $c \equiv -ab \pmod{|\nu|}$ and, hence,

$$u_{n+3} \equiv a \cdot u_{n+2} + b \cdot u_{n+1} - ab \cdot u_n \pmod{|\nu|}.$$

Now, using mathematical induction with respect to k, we get (14).

Theorem 2.5: Let $\mathbf{u} = \{u_n\} \in U_3$, $\mathbf{u}$ satisfying (4), (5), (7), and (10). Then

$$(u_2, u_j) = |u_{(2, j)}| \quad \text{for all } j \geqslant 1.$$

Proof: Let $j \geqslant 1$ be even; then, from Lemma 2.4, we get

$$(u_2, u_j) = |\nu| = |u_{(2, j)}|.$$

Now, let $j \geqslant 1$ be odd; then, from (4) and (7), it follows that $(\nu, b^{k-1} \cdot \mu) = 1$ for all $k \geqslant 1$ and, hence, from Lemma 2.4, we get

$$(u_2, u_j) = 1 = |u_{(2, j)}|.$$

3. A SPECIAL CASE OF $u_2 \neq 0$, $u_3 \neq 0$

Let $\overline{U}_3$ denote the system of all the sequence from U_3 satisfying the conditions,

$$(u_i, u_j) = |u_{(i, j)}| \quad \text{for } 1 \leqslant i, j \leqslant 6, \tag{15}$$

$$b + f = 0, \tag{16}$$

where f is the integer from (5). Further, let

$$\mathbf{c} = \{1, 2, 1, 0, 1, 2, 1, 0, \ldots\}, \quad \mathbf{d} = \{1, -2, 1, 0, 1, -2, 1, 0, \ldots\}.$$

The following theorem will give a complete characterization of all the strong divisibility sequences in $\overline{U}_3$, showing that **c** and **d** are the only strong divisibility sequences in $\overline{U}_3$, i.e., $\overline{U}_3 \cap D = \{\mathbf{c}, \mathbf{d}\}$.

Theorem 3.1: Let $\mathbf{u} = \{u_n\} \in \overline{U}_3$. Then $\mathbf{u} \in D$ if and only if $\mathbf{u} = \mathbf{c}$ or $\mathbf{u} = \mathbf{d}$.

Proof: Obviously, $\mathbf{c}, \mathbf{d} \in \overline{U}_3 \cap D$. Conversely, let $\mathbf{u} \in \overline{U}_3$ be a strong divisibility sequence. Let us denote $x = \nu \cdot (\nu f - \mu a) + \mu b$, $y = \nu^2 af + \nu\mu(f - a^2) + \mu a(b - \mu)$. Then, from (16), (6), (9), and (12), we get $\mu = \pm 1$, $x = \pm 1$, $y = \pm\nu$, so that we have eight possibilities:

(i) $\mu = 1$, $x = 1$, $y = \nu$

From $\mu = 1$ and $x = 1$, we get $b - 1 = \nu a - \nu^2 f$. Then, from $y = \nu$, we get $\nu f = \nu$ so that $f = 1$ and, consequently, $b = -1$, $a\nu = \nu^2 - 2$, and $c = \nu - a$, using (5). Then $\mathbf{u} = \{1, \nu, 1, 0, 1, \nu, \nu^2 - 3, \ldots\}$. But from $(u_4, u_7) = |u_1|$, we get $\nu = \pm 2$ and, hence, $\mathbf{u} = \mathbf{c}$ or $\mathbf{u} = \mathbf{d}$.

(ii) $\mu = 1$, $x = 1$, $y = -\nu$

Similarly, as in (i), we get $f = -1$, $b = 1$, $a = -\nu$, and $c = 0$. Then we obtain $\mathbf{u} = \{1, \nu, 1, 0, 1, -\nu, \nu^2 + 1, \ldots\}$, a contradiction, since $(u_4, u_7) = \nu^2 + 1 \neq |u_1|$.

(iii) $\mu = 1$, $x = -1$, $y = \nu$

Using $\mu = 1$, $f = -b$ in $x = -1$, we get $\nu a = -\nu^2 b + b + 1$ and then, from $y\nu = \nu^2$, we get $b \cdot (\nu^2 - 2) = \nu^2 + 2$. Let $|\nu| \geqslant 2$, then $\nu^2 \equiv -2 \pmod{(\nu^2 - 2)}$. Trivially, $\nu^2 \equiv 2 \pmod{(\nu^2 - 2)}$, so that $(\nu^2 - 2) | 4$ and, consequently, $\nu = \pm 2$. But $\nu = \pm 2$ implies $b = 3$, $a = \mp 4$, and $c = \mp 2$, a contradiction, since $(u_4, u_7) = 11 \neq |u_1|$. The remaining cases $\nu = \pm 1$ lead to $b = -3$, $a = \pm 1$, and $c = \pm 2$, a contradiction, since $(u_4, u_7) = 4 \neq |u_1|$.

(iv) $\mu = 1$, $x = -1$, $y = -\nu$

Similarly, as in (iii), we get $\nu a = -\nu^2 b + b + 1$ and $b \cdot (\nu^2 - 2) = -\nu^2 + 2$ so that $b = -1$, $a = \nu$, and $c = 0$. Then $\mathbf{u} = \{1, \nu, 1, 0, -1, -\nu, -\nu^2 + 1, \ldots\}$, a contradiction, since $(u_4, u_7) \neq |u_1|$.

(v) $\mu = -1$, $x = 1$, $y = \nu$

Similarly, as in (i), we get $f = -1$, $b = 1$, $c = a - \nu$, and $a\nu = \nu^2 + 2$, which gives $\mathbf{u} = \{1, \nu, -1, 0, 1, \nu, \nu^2 + 3, \ldots\}$, a contradiction, since $(u_4, u_7) = \nu^2 + 3 \neq |u_1|$.

(vi) $\mu = -1$, $x = 1$, $y = -\nu$

In the same way as in (i), we get $f = 1$, $b = -1$, $a = -\nu$, and $c = 0$ so that $\mathbf{u} = \{1, \nu, -1, 0, 1, -\nu, \nu^2 - 1, \ldots\}$, a contradiction, since $(u_4, u_7) = \nu^2 - 1 \neq |u_1|$.

(vii) $\mu = -1$, $x = -1$, $y = \nu$

Similarly, as in (iii), we get $b \cdot (\nu^2 + 2) = -\nu^2 + 2$ and, hence, $\nu^2 \equiv 2 \pmod{\nu^2 + 2}$. Trivially, $\nu^2 \equiv -2 \pmod{\nu^2 + 2}$, so that we get $(\nu^2 + 2) | 4$ and, consequently, $\nu^2 = -1, 0, 2$, a contradiction.

(viii) $\mu = -1$, $x = -1$, $y = -\nu$

Similarly, as in (iii), we get $\nu a = \nu^2 b + b - 1$ and $b(\nu^2 + 2) = \nu^2 + 2$, so that $b = 1$, $a = \nu$, $c = 0$. Hence, $\mathbf{u} = \{1, \nu, -1, 0, -1, -\nu, -\nu^2 - 1, \ldots\}$, a contradiction, since $(u_4, u_7) = \nu^2 + 1 \neq |u_1|$.

Remark: We did not use conditions (8), (11), and (13) in the proof of Theorem 3.1, so that we can, in fact, weaken the assumptions (15) by omitting

$$(u_3, u_5) = |u_1|, \quad (u_3, u_6) = |u_3|, \quad \text{and} \quad (u_5, u_6) = |u_1|.$$

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