STROEKER'S EQUATION AND FIBONACCI NUMBERS

ANDRZEJ MĄKOWSKI
University of Warsaw, 00-901 Warsaw, Poland

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R. J. Stroeker [1] considered the Diophantine equation

$$(x^2 + y)(x + y^2) = N(x - y)^3, \tag{1}$$

where N is a positive integer. He found all solutions of (1) for $N \leqslant 51$ and proved that if x, y satisfy this equation with $N \neq 1, 2, 4$ then

$$\max(|x|, |y|) < N^3 \quad \text{(see Theorem 1 of [1]).}$$

For every N equation, (1) has the trivial solution $x = y = -1$. Theorem 2 of [1] asserts that for odd $N > 1$ there exists a nontrivial solution with $xy \neq 0$, and for infinitely many such values of N there are at least five such solutions. The table given at the end of [1] shows that for many even N there is only the trivial solution.

Below, we exhibit a connection between (1) and Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. The following identities are well known:

$$F_k F_{k+1} + (-1)^k = F_{k-1}F_{k+2}; \tag{2}$$
$$F_{k-n}F_{k+n} - F_k^2 = (-1)^{k+n+1}F_n^2. \tag{3}$$

When we put $n = 1$ or 2 in identity (3), it becomes, respectively,

$$F_{k-1}F_{k+1} - F_k^2 = (-1)^k, \tag{4}$$
$$F_k^2 - (-1)^k = F_{k-2}F_{k+2}. \tag{5}$$

Taking (4) with k replaced by $k + 1$ and multiplying it by F_{k+1}, we get

$$F_k F_{k+1} F_{k+2} - F_{k+1}^3 = (-1)^{k+1} F_{k+1},$$
$$F_{k+1}^3 - (-1)^k (F_{k+2} - F_k) = F_k F_{k+1} F_{k+2},$$
$$F_{k+1}^3 + (-1)^k F_k = [F_k F_{k+1} + (-1)^k] F_{k+2},$$

which, in view of (2), may be written in the form

$$F_{k+1}^3 + (-1)^k F_k = F_{k-1} F_{k+2}^2. \tag{6}$$

Multiplying (5) and (6), we get

$$[F_k^2 - (-1)^k][F_{k+1}^3 + (-1)^k F_k] = F_{k-2}F_{k-1}(F_k + F_{k+1})^3,$$

and

$$[F_k^2 F_{k+1}^2 - (-1)^k F_{k+1}^2][F_{k+1}^4 + (-1)^k F_k F_{k+1}] = F_{k-2} F_{k-1} (F_k F_{k+1} + F_{k+1}^2)^3.$$

This shows that, for $N = F_{k-2}F_{k-1}$, equation (1) is satisfied by

$$x = F_k F_{k+1},\ y = -F_{k+1}^2 \quad (k \text{ even})$$

and by

$$x = F_{k+1}^2,\ y = -F_k F_{k+1} \quad (k \text{ odd}).$$

Therefore, for infinitely many values of N, the number $\max(|x|, |y|)$ is larger than $11N$ because

$$\lim_{k\to\infty} \frac{F_{k+1}^2}{F_{k-2}F_{k-1}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^5 > 11.$$

Furthermore, since there are infinitely many even Fibonacci numbers, there are infinitely many positive even integers N such that (1) has a nontrivial solution. The last result, however, can be proved in a simpler way:

For $N = [(a + 1)^3 + 1](a^3 + 1)$, the numbers $x = a(a + 1)^2$, $y = a^2(a + 1)$ satisfy (1).

The following question remains open: Do there exist infinitely many positive (even) integers N such that equation (1) has only the trivial solution?

REFERENCE

1. R. J. Stroeker. "The Diophantine Equation $(x^2 + y)(x + y^2) = N(x - y)^3$." *Simon Stevin* 54 (1980):151-163.

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