CONVOLUTION TREES AND PASCAL-T TRIANGLES

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(Submitted December 1986)

1. INTRODUCTION

Pascal (1623-1662) made extensive use of the famous arithmetical triangle which now bears his name. He wrote upon its properties in 1653, but the paper was not printed until 1665 ([1], "Traité du triangle arithmétique"). The triangle now appears in virtually every text on elementary combinatorics. All textbook authors note the recurrence relation satisfied by binomial coefficients in adjacent rows of the triangle, and a few point out the "curious" fact that certain diagonals of the triangle have Fibonacci numbers as their sums (apparently first noted by E. Lucas in 1876).

In this paper we give a graph theory approach that provides an easy access to associations between Pascal-T triangles and generalized Fibonacci sequences. The approach is to use certain sequences of tree graphs, which are called *convolution trees* for a reason which is explained in Section 3. These trees consist of nodes and branches that are introduced and "grown" according to a given construction rule; integer weights are assigned to the nodes as the construction proceeds.

The weights are obtained from a *color sequence* $\{c_n\}$, and they are assigned to the nodes in a well-defined manner. The choice of generalized Fibonacci sequences of use for $\{c_n\}$ enables many attractive identities to be discovered, almost by inspection.

In Section 6 we define a *level counting function* for the trees that counts certain of the colored nodes in the trees and also provides generalizations of Pascal's triangle. The arithmetic triangles which arise are known as Pascal-T triangles [2].

The main results of the paper are collected together as Theorem 5 in Section 6. This demonstrates the links between various properties of the Pascal-T triangles and the generalized Fibonacci sequences which the study of colored convolution trees reveals.

A *graph* is a set of *nodes* (or *points*) together with a set of *edges* (in tree graphs they are often called *branches*). An *edge* is, informally, a line joining two of the nodes. The total number of edges which attach to a given node is the *valency* (or *degree*) of that node. A *circuit* is a *path* in a graph which proceeds from node-edge-node-edge-node-$\cdots$-node and is such that the first node and the last node are the same node.

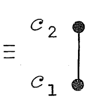
A *tree* is a graph that has no circuits.

In a tree we may distinguish any one node and call it the *root* of the tree. Then we may distinguish all nodes in the tree (other than the root) whose valencies are one (unity) and call them *leaf nodes*.

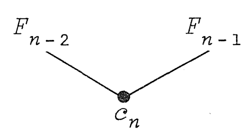
We are now in a position to present the rules by which colored convolution trees are constructed.

2. FIBONACCI CONVOLUTION TREES

The Fibonacci convolution trees are defined by a *recurrence construction* which builds the trees $\{F_n\}$ sequentially, assigning the integer weights or *colors* $\{c_n\}$ as they are built. A similar construction (but not the coloring) was given in [3]. The method parallels the definition of Fibonacci numbers (namely $f_n = f_{n-2} + f_{n-1}$, with $f_1 = 1$, $f_2 = 1$), with a binary operation $\oplus$ that works as follows. We define the initial colored rooted trees in the sequence to be

$F_1 \equiv c_1$ • and $F_2 \equiv$ 

Then, given any two consecutive trees F_{n-2}, F_{n-1}, we obtain the next tree by $F_n = F_{n-2} \oplus F_{n-1}$, the joining operation $\oplus$ being indicated by the diagram:



$\equiv F_n.$

Note that one new root node, labelled c_n, is introduced during this operation. Figure 1 shows the first four trees in the sequence. In Figure 1 and in subsequent tree diagrams, the color alone is used to depict the colored node, for convenience.

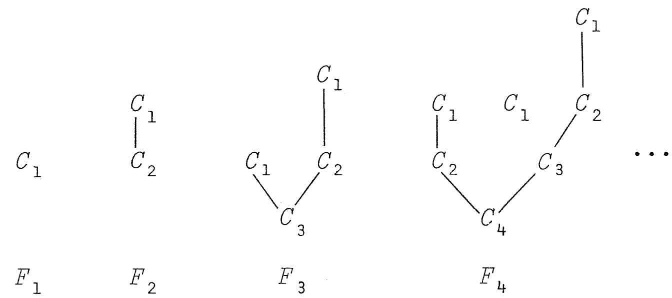


Figure 1. Fibonacci Convolution Tree Sequence

3. PROPERTIES OF A CONVOLUTION TREE

We next tabulate basic graph properties of the convolution trees. It will be seen that the parameters listed have an attractive set of formulas in terms of the Fibonacci numbers $\{f_n\} = \{1, 1, 2, 3, 5, \ldots\}$. Some graph terms used in the table may require definition for the reader, thus:

> In any rooted tree a unique path may be traced from the root to any other given node in the tree. The number of edges (branches) in that path is called the *level* of the given node. The *height* of a convolution tree is the maximum level occurring.

The symbols $(\mathbf{c} * \mathbf{f})_n$ refer to the n^{th} term of the convolution of sequences **c** and **f**; this term is defined to be $c_1 f_n + c_2 f_{n-1} + \cdots + c_n f_1$.

Table 1. Properties of Fibonacci Convolution Trees

	Parameter		Formula (for F_n)
(i)	Number of nodes		$F^n \equiv \sum_1^n f_i$
(ii)	Number of edges		$F^n - 1$
(iii)	Number of nodes	$v = 1$	f_n
	of valency v:	$v = 2$	$f_{n-1} + 1$
	$(n > 2)$	$v = 3$	$f_n - 2$
(iv)	Number of leaf nodes		f_n
(v)	Height		$n - 1$
(vi)	Weight (sum of node colors)		$(\mathbf{c} * \mathbf{f})_n$
(vii)	Lowest leaf-node level		$\left[\frac{n}{2}\right]$
(viii)	Number of leaf nodes at level m		$\binom{m}{n - m - 1}$

Proofs: All of the formulas given in the table can be proved using a combination of graph definitions, the tree construction rule, simple algebra, and mathematical induction.

The convolution result (vi) is the reason for the name we gave to the tree graphs. To demonstrate a proof method, we shall give the proof for (vi) only. It is proved as follows: using $\Omega(F)$ to mean "weight of F" (i.e., the sum of the node colors in F), we have, from the construction rule,

$$\Omega(F_n) = \Omega(F_{n-2}) + \Omega(F_{n-1}) + c_n, \text{ for } n \geqslant 2. \tag{1}$$

Noting that $\Omega(F_1) = c_1 f_1 = (\mathbf{c} * \mathbf{f})_1$, and $\Omega(F_2) = c_1 f_2 + c_2 f_1 = (\mathbf{c} * \mathbf{f})_2$, it is easy to proceed by induction. That is, we may show that, if

$$\Omega(F_i) = c_1 f_i + c_2 f_{i-1} + \cdots + c_i f_1 = (\mathbf{c} * \mathbf{f})_i$$

for $i = 1, 2, \ldots, n$, then

$$\Omega(F_{n+1}) = (\mathbf{c} * \mathbf{f})_{n+1}.$$

We leave the details to the reader.

4. SOME THEOREMS DERIVED FROM THE TREES

Weighted convolution trees are structured configurations of integers, and in the long tradition of such structures (c.f. figurate numbers, Ferrer's diagrams and the like) they can be used to reveal identities and relations between given sequence elements. The next four theorems illustrate many interesting relations between Fibonacci numbers, Fibonacci convolutions, and binomial coefficients.

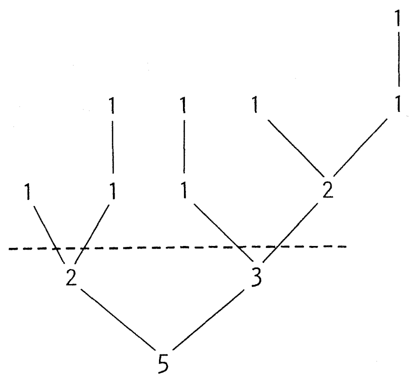
Theorem 1 (Lucas, 1876): $f_n = \sum_m \binom{m}{n-m-1}$ with m varying from $\left[\frac{n}{2}\right]$ to $n-1$, where $[x]$ is the greatest integer function.

This follows from formulas (iv) and (viii) of Table 1.

Theorem 2: Let $r = \left[\frac{n-1}{2}\right]$ with $n \geqslant 3$. Then

$$rf_n = (\mathbf{f} * \mathbf{f})_n - \sum_{i=0}^{r} \binom{r}{i} (\mathbf{f} * \mathbf{f})_{i+1}.$$

Proof strategy: This theorem gives a relationship between Fibonacci integers, terms of the convolution sequence $\mathbf{f} * \mathbf{f}$, and binomial coefficients. It is an example of how interesting identities may be discovered virtually by inspection of the colored convolution trees. We shall describe the proof strategy with reference to tree F_5. The reader may care to fill in the details of the proof, and then to look for other identities of a similar nature.



Tree F_5

First we note that a cut along a dotted line drawn immediately below the lowest leaf-node (which is $[n/2]$; see Table 1(vii)) would, in effect, split the tree into a lower portion that is a full binary-tree and an upper collection of separated smaller convolution trees.

By *full binary tree* we mean a rooted tree of which the root node has valency two, and all other non-leaf nodes have valency three.

Next we observe that the smaller convolution trees are F_1, F_2, and F_3 and that they occur with frequencies given by the binomial coefficients

$$\binom{r}{0},\ \binom{r}{1},\ \text{and } \binom{r}{2},\ \text{with } r = \left[\frac{n}{2}\right] = 2.$$

Collecting this information together, and equating the weight of F_5 to the sum of the weights of all the subtrees we have described, we get

$$\Omega(F_5) = (\mathbf{f} * \mathbf{f})_5 = \Omega(\text{full binary tree}) + \sum_{i=0}^{2} \binom{2}{i} (\mathbf{f} * \mathbf{f})_{i+1}.$$

Finally, inspection of the full binary tree reveals that the sum of the colors on the nodes at each level is f_5; and there are $r = 2$ levels, so

$$\Omega(\text{full binary tree}) = 2f_5.$$

Inserting this in the above equation and rearranging to place $2f_5$ alone on the left-hand side, we obtain a demonstration of the formula for the tree F_5.

Each one of the observations made with regard to the properties of the subtrees of F_5 can be shown by induction to hold, generally, for subtrees obtained similarly from tree F_n. Then the proof strategy carries through for F_n, for $n \geqslant 3$.

Note that the Lucas sum for f_n from Theorem 1 can be exchanged for f_n in Theorem 2 and another identity obtained immediately.

Theorem 3 (general **c**): We have already noted in Section 3 the *fundamental convolution property*, namely,

$$(\mathbf{c} * \mathbf{f})_n = (\mathbf{c} * \mathbf{f})_{n-2} + (\mathbf{c} * \mathbf{f})_{n-1} + c_n,$$

where **f** is the Fibonacci sequence and $\mathbf{c} = \{c_1, c_2, \ldots\}$.

We now examine the effect on the total weight, say $\Omega_n(\mathbf{c})$, of the n^{th} convolution tree when $\mathbf{c}$ is changed to $\mathbf{c}^{(r)} = \{c_{r+1}, c_{r+2}, \ldots\}$. In terms of the shift operator E, operating on the subscripts of the sequence terms c_i, we can write $\mathbf{c}^{(1)} = E\mathbf{c}$; and, in general, $\mathbf{c}^{(r)} = E^r\mathbf{c} = \{c_{r+1}, \ldots\}$. Let us also introduce the difference operator Δ, now operating on subscripted terms, so that $\Delta\mathbf{c} = \{c_2 - c_1, c_3 - c_2, \ldots\}$; and then $\Delta^2\mathbf{c} = \Delta(\Delta\mathbf{c})$, and so on to $\Delta^r\mathbf{c}$ in general. Then the following results hold, pertaining to the total weight of the convolution trees. We now give Theorem 4 as further illustration of how attractive identities and formulas (this time involving E and Δ) can be derived with little effort from the colored tree sequence.

Theorem 4:

(i) $\delta_n^{(1)} \equiv \Omega_n(E\mathbf{c}) - \Omega_n(\mathbf{c}) = (\mathbf{f} * \Delta\mathbf{c})_n$;

$\vdots$

$\delta_n^{(r)} \equiv \Omega_n(E^r\mathbf{c}) - \Omega_n(\mathbf{c}) = (\mathbf{f} * E^{r-1}(\Delta\mathbf{c}))_n + \sum_{j=1}^{r-1} \delta_n^{(j)}, \quad r \geqslant 2.$

(ii) (setting $\mathbf{c} = \mathbf{f}$)

(a) $\Delta\mathbf{c} = \Delta\mathbf{f} = E^{-1}\mathbf{f}$; $(\mathbf{f} * \Delta^r\mathbf{f})_n = (\mathbf{f} * E^r\mathbf{f})_{n-r}$.

(b) $(\mathbf{f} * \mathbf{f})_n = f_n + (\mathbf{f} * E\mathbf{f})_{n-1}$.

(c) $\Omega_n(E^r\mathbf{f}) = \Omega_n(E^{r-2}\mathbf{f}) + \Omega_n(E^{r-1}\mathbf{f})$, $r > 2$, with

$\Omega_n(E^r\mathbf{f}) = (\mathbf{f} * \mathbf{f})_n$ when $r = 0$, and

$= (\mathbf{f} * \mathbf{f})_n + (\mathbf{f} * \mathbf{f})_{n-1}$ when $r = 1$.

(iii) [corollary of (ii)(c), writing $\Omega_{n,r}$ for $\Omega_n(E^r\mathbf{f})$]

$$\Omega_{n,r} = (\mathbf{f} * \mathbf{f})_n f_{r+1} + (\mathbf{f} * \mathbf{f})_{n-1} f_r, \quad r \geqslant 1.$$

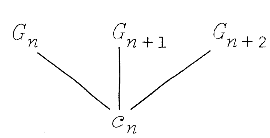
The proofs of (i), (ii), and (iii) require only simple algebra and Fibonacci number identities.

5. HIGHER ORDER CONVOLUTION TREES

The construction rules given in Section 2 may be extended to define sequences of higher-order convolution trees. Thus, for third-order trees:

Recurrence rule: $G_{n+3} = G_n \oplus G_{n+1} \oplus G_{n+2}$, using a triple fork to effect the tree combinations thus:

$G_{n+3} \equiv$



In Figure 2 we show the first five trees in the sequence obtained when the F_1, F_2, F_3 trees are used as the initial ones.

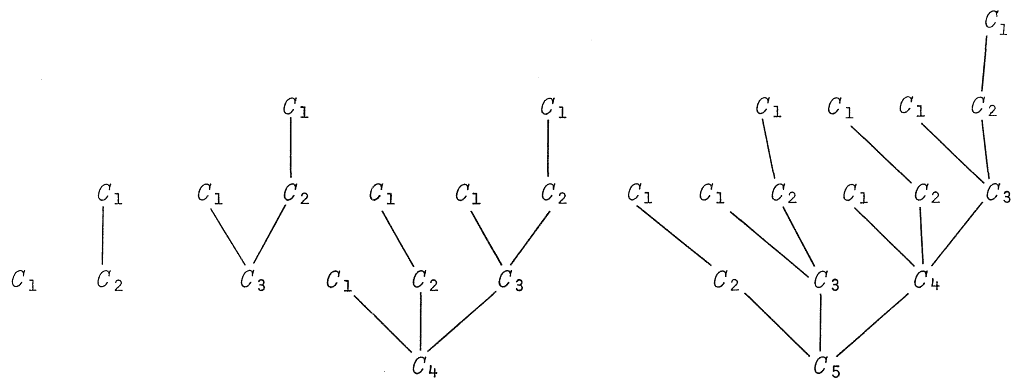


Figure 2. The First Five Third-Order Convolution Trees

We will not tabulate their structural properties as we did for the second-order ones, but we may note that the numbers of leaf nodes follow the sequence $\mathbf{g} = \{1, 1, 2, 4, 7, \ldots\}$, and that the weight $\Omega(G_n)$ can be shown to be $(\mathbf{c} * \mathbf{g})_n$, which are generalizations of the second-degree convolution tree properties.

We are now in a position to derive Pascal-T triangles from the sequences of trees.

6. A COMBINATORIC FUNCTION AND THE PASCAL-T TRIANGLES

Consider the convolution tree G_n, colored by integers of the sequence $\mathbf{c} = \{c_1, c_2, c_3, \ldots\}$. We define the *level counting function*:

$$L \equiv \binom{n}{m|i} \equiv \text{the number of nodes in } G_n \text{ having level } m \text{ and color } c_i.$$

Then, if G is defined in some tree sequence $\{G_n : n = 1, 2, 3, \ldots\}$, we can tabulate L in a sequence of (m, n) tables for each value of i. We show tables for the second- and third-order trees with regard to color c_1 only.

Table 2. $\binom{n}{m|1}$ for the Second-Order Trees F_n

m \ n	F_1	F_2	F_3	F_4	F_5	F_6	F_7	Row Sum
0	1	0	0	0	0	0	0 ...	1
1	0	1	1	0	0	0	0 ...	2
2	0	0	1	2	1	0	0 ...	4
3	0	0	0	1	3	3	1 ...	8
4	0	0	0	0	1	4	6 ...	(16)
5	0	0	0	0	0	1	5 ...	(32)
6	0	0	0	0	0	0	1 ...	(64)
Column Sum	1	1	2	3	5	8	13 ...	

We observe the following:

(i) the nonzero elements correspond to Pascal's triangle, the rows beginning on the diagonal; let us designate this triangle $\Delta^{(2)}$;

(ii) the m^{th} row sum of the table is 2^m;

(iii) the j^{th} column sum of the table is f_j, the j^{th} Fibonacci number.

Table 3. $\binom{n}{m|1}$ for the Third-Order Trees G_n

$m \backslash n$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	Row Sum
0	1	0	0	0	0	0	0 ...	1
1	0	1	1	1	0	0	0 ...	3
2	0	0	1	2	3	2	1 ...	9
3	0	0	0	1	3	6	7 ...	(27)
4	0	0	0	0	1	4	10 ...	(81)
5	0	0	0	0	0	1	5 ...	(243)
6	0	0	0	0	0	0	1 ...	(729)
Column Sum	1	1	2	4	7	13	24 ...	

Notes:

(i) the triangle now resting on the leading diagonal is the third-degree one, $\Delta^{(3)}$;

(ii) the m^{th} row sum of the table is 3^m;

(iii) the j^{th} column sum of the table is g_j, where g is defined by

$$g_{n+3} = g_n + g_{n+1} + g_{n+2},$$

with $(g_1, g_2, g_3) = (f_1, f_2, f_3)$, a generalized Fibonacci sequence.

It should be clear from the construction rules given in Section 5 how we can extend the order of convolution trees indefinitely, obtaining the sequence $\{G_2\}$, $\{G_3\}$, $\{G_4\}$, ... of tree sequences. Then, tabulating $\binom{n}{m|1}$ for each would give a sequence of the triangles $\Delta^{(\delta)}$, $\delta = 2, 3, 4, \ldots$; and the row and column sums of the tables would be, respectively, powers of δ and generalized Fibonacci numbers.

We note also that every $\binom{n}{m|1}$ is a multinomial coefficient; it is easy to show that the m-row elements in each table are generated by the function:

$$x(x + x^2 + x^3 + \cdots + x^\delta)^m,$$

where δ is the order of the trees being considered.

We show below the second-, third-, and fourth-order triangles in the form that Pascal's triangle is usually shown. We do this in order to comment on the generalized row-to-row method of constructing the elements.

$$
\begin{array}{ccccccccccc}
 & & & & & \Delta^{(2)} & & & & & \\
 & & & & & 1 & & & & & \\
 & & & & 1 & 2 & 1 & & & & \\
 & & & 1 & 3 & & 3 & 1 & & & \\
 & & 1 & 4 & & 6 & & 4 & 1 & & \\
 & 1 & 5 & & 10 & & 10 & & 5 & 1 & \\
 & & \cdots & & & & & \cdots & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \Delta^{(3)} & & & & \\
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
 & \cdots & & & \cdots & & & \cdots &
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & \Delta^{(4)} & & & & & & \\
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & \\
 & 1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 & & \\
1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \\
 & \cdots & & & \cdots & & & & \cdots & & & \cdots &
\end{array}
$$

Figure 3. Pascal-T Triangles

Note that, in each case, to get the j^{th} element in the m^{th} row, take the sum of the δ ($\delta = 2, 3, 4$) elements immediately above it in the preceding [i.e., the $(m-1)^{\text{th}}$] row. Use zeros if the summation has to extend beyond a boundary of the triangle. For example, to get 10, the third element in row 5 of $\Delta^{(4)}$, we add 0 + 1 + 3 + 6.

Theorem 5 (Pascal-Lucas-Turner): Let S_δ be a sequence of colored convolution trees of order δ, $\delta = 2, 3, 4, \ldots$. Then the level function $\binom{n}{m|i}$, with $i = 1$, has a table of values with the following properties:

(i) $m = 0, 1, 2, \ldots;\quad n = 1, 2, 3, \ldots;$

(ii) the leading diagonal elements are all 1's, and elements below this diagonal are all 0's;

(iii) the sum of the m-row elements is δ^m;

(iv) the sum of the n-column elements is g_n, where g is the generalized Fibonacci sequence defined by

$$g_{n+\delta} = \sum_{i=0}^{\delta-1} g_{n+i}, \text{ with initial values } f_1, f_2, \ldots, f_\delta;$$

(v) $\binom{n}{m|1}$ is the coefficient of x^n in the expansion of $x\left(\sum_{i=1}^{\delta} x^i\right)^m$;

(vi) $\binom{n}{m|1} = \binom{n-1}{m-1|1} + \binom{n-2}{m-1|1} + \cdots + \binom{n-\delta}{m-1|1}$ for $n > 1$, $m > 0$; with

$$\binom{1}{0|1} = 1, \binom{n}{0|1} = 0, \text{ for } n > 1, \text{ and } \binom{n-i}{m-1|1} \equiv 0 \text{ when } n < i.$$

Proofs: The proofs follow directly from the recurrence construction rules for the trees.

7. OTHER LEVEL-FUNCTION TRIANGLES

Although we have presented our topic so far by showing how level functions (with $i = 1$) provide Pascal's triangle and generalizations of it, we would now like to shift the point of view firmly.

In the theory of convolution trees, the level function seems to us to be an important object of study. Every sequence of convolution trees gives rise to a sequence of tables for the level functions $\binom{n}{m|i}$, and the types of values they take depend entirely on the construction rules used to define the trees. Changing the tree recurrences, or the initial trees, or using a more complex coloring rule, will produce triangles of numbers which are not, in general, multinomial coefficients. If generating functions can be found, they will be more complex than the ones given above.

Therefore, we wish to view the tabulation of level functions of convolution trees as a broad topic in its own right. Pascal's triangle arises as a special case in connection with second-order Fibonacci trees.

For reasons of space we cannot give many examples of other triangles here; however, we discuss two further cases to help make our point clear. The first gives rise to "shifted" Pascal triangles; the second arises from Lucas trees, and turns out to be a superposition of two Pascal triangles.

Case 1. $\binom{n}{m|2}$, from the Fibonacci trees

If we look at the rooted trees in $\{F_i\}$ and $\{G_i\}$, we see that all the leaf nodes are colored c_1. Pruning any tree F_n (i.e., removing all the leaf nodes and their adjacent branches) leaves the tree F_{n-1}, but with colors c_2, c_3, c_4, ... instead of c_1, c_2, c_3, ...

Hence, the table of $\binom{n}{m|2}$ again has a Pascal triangle in it, but "shifted to the right" and starting at the diagonal above the leading diagonal.

Similarly, $\binom{n}{m|3}$ has a Pascal triangle shifted one step further to the right; and so on.

Case 2. $\binom{n}{m|1}$, **Lucas convolution trees**

Using a special initial tree, L_2, we can generate the sequence L_1, L_2, L_3, ..., called Lucas convolution trees and shown below in Figure 4. Note that the numbers of leaves follow the Lucas sequence $\ell = 1, 3, 4, 7, \ldots$, which is generated by the recurrence equation $\ell_{n+2} = \ell_n + \ell_{n+1}$, with $\ell_1 = 1$, $\ell_2 = 3$. The color sequence used is $\mathbf{c} = (c_0, c_1, c_2, c_3, \ldots)$; the recurrence construction begins with tree L_3 and color c_4.

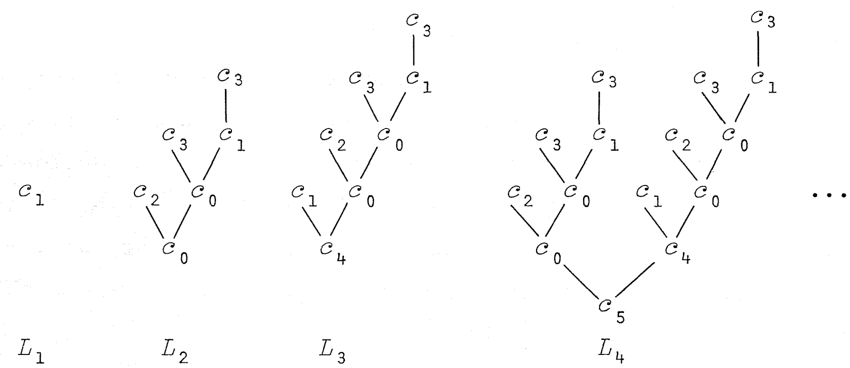


Figure 4. The Lucas Convolution Trees

These trees have many properties which relate the Fibonacci and Lucas numbers. We give the table for $\binom{n}{m|1}$, then follow it by the Lucas-T triangle for this level function.

Table 4. $\binom{n}{m|1}$ for the Second-Order Lucas Trees

$m \backslash n$	L_1	L_2	L_3	L_4	L_5	L_6	L_7	Row Sum
0	1	0	0	0	0	0	0 ...	1
1	0	0	1	0	0	0	0 ...	1
2	0	1	0	1	1	0	0 ...	$3 \times 2^0 = 2^0 + 2^1$
3	0	0	1	1	1	2	1 ...	$3 \times 2^1 = 2^1 + 2^2$
4	0	0	0	1	2	2	3 ...	$3 \times 2^2 = 2^2 + 2^3$
5	0	0	0	0	1	3	4 ...	$3 \times 2^3 = 2^3 + 2^4$
6	0	0	0	0	0	1	4 ...	$3 \times 2^4 = 2^4 + 2^5$
Column Sum	1	1	2	3	5	8	...	

Note that the row sums are (after $m = 1$) expressible as $2^{m-2} + 2^{m-1}$, and that the column sums are again Fibonacci numbers. The diagram below shows (by dotted and full lines) how the triangle from these Lucas trees is the superposition of two Pascal triangles (after $m = 0$).

```
                  1
                0   1
          1   0   1   1
        1   1   1   2   1
      1   2   2   3   3   1
    1   3   4   5   6   4   1
  1   4   7   9  11  10   5   1
     ...        ...       ...
```

Figure 5. The Lucas $\binom{n}{m|1}$ Triangle

We have developed a notation for writing the $\binom{n}{m|i}$ triangles to be derived from various types of recurrently constructed and colored trees, expressing them as superpositions of triangles of multinomial coefficients. The formulas can be given once the construction and coloring rules are given.

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