PROPERTIES OF A RECURRING SEQUENCE

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1. Introduction

Recurring sequences such as the Fibonacci sequence defined by

$$F_0 = 0,\ F_1 = 1;\ F_n = F_{n-1} + F_{n-2},\ n \ge 2 \tag{1.1}$$

and the Lucas sequence given by

$$L_0 = 2,\ L_1 = 1;\ L_n = L_{n-1} + L_{n-2},\ n \ge 2, \tag{1.2}$$

have been extensively studied because they have many interesting combinatorial properties.

In the present paper, we study the sequence

$$\{L_{2n+1}\}_{n=0}^{\infty},$$

which obviously satisfies the recurrence relation

$$L_1 = 1,\ L_3 = 4,\ 3L_{2n+1} - L_{2n-1} = L_{2n+3}, \tag{1.3}$$

and is generated by [9, p. 125]

$$\sum_{k=0}^{n} L_{2n+1}t^n = (1 + t)(1 - 3t + t^2)^{-1},\ |t| < 1. \tag{1.4}$$

It can be shown that these numbers possess the following interesting property,

$$\sum_{n=0}^{\infty} (-1)^{n+k}\binom{2n+1}{n-k} L_{2k+1} = 1, \tag{1.5}$$

for every nonnegative integral value of n, which can be rewritten as

$$\sum_{k=0}^{n} \frac{(-1)^k L_{2k+1}}{(n-k)!(n+k+1)!} = \frac{(-1)^n}{(2n+1)!}. \tag{1.6}$$

In sections 2 and 3, we study two different q-analogues of L_{2n+1}. In the last section we pose some open problems and make some conjectures. As usual, we shall denote the rising q-factorial by

$$(a;\ q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}. \tag{1.7}$$

Note that, if n is a positive integer, then

$$(a;\ q)_n = (1 - a)(1 - aq)\ \ldots\ (1 - aq^{n-1}), \tag{1.8}$$

and

$$\lim_{n\to\infty}(a;\ q)_n = (a;\ q)_\infty = (1 - a)(1 - aq)(1 - aq^2)\ldots\ . \tag{1.9}$$

The Gaussian polynomial $\begin{bmatrix} n \\ m \end{bmatrix}$ is defined by [4, p. 35]

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} (q;\ q)_n/(q;\ q)_m(q;\ q)_{n-m} & \text{if } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

2. First q-Analogue of L_{2n+1}

To obtain our first q-analogue of L_{2n+1}, we use the following lemma, due to Andrews [5, Lemma 3, p. 8].

Lemma 2.1: If, for $n \ge 0$,

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q;\ q)_{n-k}(aq;\ q)_{n+k}}, \tag{2.1}$$

then

$$\alpha_n = (1 - aq^{2n}) \sum_{k=0}^{n} \frac{(aq;\ q)_{n+k-1}(-1)^{n-k}q^{\binom{n-k}{2}}\beta_k}{(q;\ q)_{n-k}}. \tag{2.2}$$

Multiplying both sides of (2.1) by $(1 - q)^{-1}$, with $a = q$ and

$$\beta_n = \frac{(-1)^n}{(q^2;\ q)_{2n}},$$

and using (1.8), we obtain

$$\frac{(-1)^n}{(q;\ q)_{2n+1}} = \sum_{k=0}^{n} \frac{\alpha_k}{(q;\ q)_{n-k}(q;\ q)_{n+k+1}},\ n \ge 0, \tag{2.3}$$

which, when compared with (1.6), will give us our first q-analogue of L_{2n+1} if we let α_k play the role of $(-1)^k L_{2k+1}$. Observe that (2.3), by using (1.10), is equivalent to

$$\sum_{k=0}^{n} (-1)^n \alpha_k \begin{bmatrix} 2n + 1 \\ n - k \end{bmatrix} = 1,\ n \ge 0. \tag{2.4}$$

Letting $\alpha_k = C_k(q)(-1)^k$ in (2.4) and (2.3), we have

$$\sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} 2n + 1 \\ n - k \end{bmatrix} C_k(q) = 1,\ n \ge 0, \tag{2.5}$$

and, by applying Lemma 2.1 to (2.3),

$$C_n(q) = \sum_{k=0}^{n} \begin{bmatrix} n + k \\ n - k \end{bmatrix} \frac{(1 - q^{2n+1})q^{\binom{n-k}{2}}}{(1 - q^{2k+1})},\ n \ge 0. \tag{2.6}$$

Now we prove the following:

Theorem 2.1: For all $n \ge 0$, $C_n(q)$ is a polynomial.

Proof: Let

$$D_{n,j}(q) = \begin{bmatrix} n + j \\ n - j \end{bmatrix} \frac{1 - q^{2n+1}}{1 - q^{2j+1}} q^{\binom{n-j}{2}}. \tag{2.7}$$

Since

$$C_n(q) = \sum_{j=0}^{n} D_{n,j}(q),$$

it suffices to prove that $D_{n,j}(q)$ is a polynomial. Now

$$\begin{aligned}
D_{n,j}(q) &= \begin{bmatrix} n+j \\ n-j \end{bmatrix} \frac{(1 - q^{2j+1} + q^{2j+1} - q^{2n+1})}{(1 - q^{2j+1})} q^{\binom{n-j}{2}} \\
&= \begin{bmatrix} n+j \\ n-j \end{bmatrix} \left(1 + \frac{q^{2j+1}(1 - q^{2n-2j})}{1 - q^{2j+1}}\right) q^{\binom{n-j}{2}} \\
&= \begin{bmatrix} n+j \\ n-j \end{bmatrix} q^{\binom{n-j}{2}} + \frac{(q;\,q)_{n+j} q^{2j+1+\binom{n-j}{2}}(1 - q^{n-j})(1 + q^{n-j})}{(q;\,q)_{n-j}(q;\,q)_{2j}(1 - q^{2j+1})} \\
&= \begin{bmatrix} n+j \\ n-j \end{bmatrix} q^{\binom{n-j}{2}} + \begin{bmatrix} n+j \\ n-j-1 \end{bmatrix} q^{2j+1+\binom{n-j}{2}}(1 + q^{n-j}),
\end{aligned}$$

which is obviously a polynomial.

Theorem 2.2: The coefficient of q^n in $C_\infty(q)$ equals twice the number of partitions of n into distinct parts.

Proof: $C_\infty(q) = \lim_{n\to\infty} C_n(q) = \lim_{n\to\infty} \sum_{j=0}^{n} \begin{bmatrix} 2n-j \\ j \end{bmatrix} \frac{(1 - q^{2n+1})}{(1 - q^{2n-2j+1})} q^{\binom{j}{2}}$

$= \sum_{j=0}^{\infty} \frac{1}{(q;\,q)_j} q^{\binom{j}{2}}$, since it can be shown that

$$\lim_{n\to\infty} \begin{bmatrix} 2n+a \\ n+b \end{bmatrix} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \tag{2.8}$$

Using the identity [4, Eq. (2.2.6), p. 19], we have

$$\sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}}{(q;\,q)_j} = \prod_{n=0}^{\infty} (1 + q^n) = 2\prod_{n=1}^{\infty}(1 + q^n). \tag{2.9}$$

Noting that $\prod_{n=1}^{\infty}(1 + q^n)$ generates partitions into distinct parts, we are done.

We now note that the numbers

$$D_{n,n-j}(1) = d_{n,j}$$

have a combinatorial meaning. However, we first recall the definitions of lattice points and lattice paths.

Definition 2.1: A point whose coordinates are integers is called a lattice point. (Unless otherwise stated, we take these integers to be nonnegative.)

Definition 2.2: By a lattice path (or simply a path), we mean a minimal path via lattice points taking unit horizontal and unit vertical steps.

In Church [2], it is shown that $d_{n,k}$ $(0 \le k \le n)$ is the number of lattice paths from $(0,\ 0)$ to $(2n + 1 - k,\ k)$ under the following two conditions:

(1) The paths do not cross $y = x + 1$ (or, equivalently, do not have two vertical steps in succession).

(2) The first and last steps cannot both be vertical.

Example: For $n = 3$, we have $d_{3,0} = 1$, $d_{3,1} = 7$, $d_{3,2} = 14$, and $d_{3,3} = 7$.

The values $d_{n,k}$ also appear along the rising diagonals (see [8, p. 486]).

3. Second q-Analogue of L_{2n+1}

The second q-analogue of the numbers L_{2n+1} is suggested by the q-extension of Fibonacci numbers found in the literature (cf. [3, p. 302; 1, p. 7]).

Equation (1.4) can be written as

$$\sum_{n=0}^{\infty} L_{2n+1}\, t^n = (1 + t) \sum_{n=0}^{\infty} \frac{t^n}{(1 - t)^{2n+2}}, \tag{3.1}$$

provided $|t/(1 - t)^2| < 1$.

Letting

$$\sum_{n=0}^{\infty} \overline{C}_n(q) t^n = (1 + t) \sum_{n=0}^{\infty} \frac{q^{n^2} t^n}{(t;\, q)_{2n+2}}, \tag{3.2}$$

we have

$$\sum_{n=0}^{\infty} \overline{C}_n(q) t^n = (1 + t) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \begin{bmatrix} 2n + 1 + m \\ m \end{bmatrix} q^{n^2} t^{n+m}, \tag{3.3}$$

by using [4, Eq. (3.3.7), p. 36], which is

$$(z;\, q)_N^{-1} = \sum_{j=0}^{\infty} \begin{bmatrix} N + j - 1 \\ j \end{bmatrix} z^j. \tag{3.4}$$

Equating the coefficients of t^n in (3.3), we get

$$\overline{C}_n(q) = \sum_{m=0}^{n} B_{n,m}(q) + \sum_{m=0}^{n-1} B_{n-1,m}(q), \tag{3.5}$$

where

$$B_{n,m}(q) = q^{(n-m)^2} \begin{bmatrix} 2n - m + 1 \\ m \end{bmatrix}. \tag{3.6}$$

Since each $B_{n,m}(q)$ is a polynomial, $\overline{C}_n(q)$ is also a polynomial for all $n \geq 0$.

Theorem 3.1: Let

$$\overline{C}_\infty(q) = \lim_{t \to 1}(1 - t) \sum_{n=0}^{\infty} \overline{C}_n(q) t^n. \tag{3.7}$$

Then

$$\overline{C}_\infty(q) = 2(P_1(q) + qP_2(q)), \tag{3.8}$$

where $P_1(q)$ is an enumerative generating function which generates partitions into parts which are either odd or congruent to 16 or 4 (mod 20), and $P_2(q)$ is another enumerative generating function which generates partitions into parts which are either odd or congruent to 12 or 8 (mod 20).

Proof: Starting with the left-hand side of (3.7), we have

$$\bar{C}_\infty(q) = \lim_{t\to 1}(1 - t)\sum_{n=0}^{\infty}\frac{(1 + t)q^{n^2}t^n}{(t;\ q)_{2n+2}} = 2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\ q)_{2n+1}}$$

$$= 2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\ q)_{2n}}\left(1 + \frac{q^{2n+1}}{1 - q^{2n+1}}\right)$$

$$= 2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\ q)_{2n}} + 2q\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q;\ q)_{2n+1}}.$$

Now, an appeal to the following two identities found in Slater's compendium [6, I-(74), p. 160; I-(96), p. 162], i.e.,

$$\prod_{n=1}^{\infty}(1 - q^{20n-8})(1 - q^{20n-12})(1 - q^{20n})$$

$$= \prod_{n=1}^{\infty}\frac{(1 - q^{2n})}{(1 + q^{2n-1})}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;\ q)_{2n}}, \tag{3.9}$$

and

$$\prod_{n=1}^{\infty}(1 - q^{10n-4})(1 - q^{10n-6})(1 - q^{20n-18})(1 - q^{20n-2})(1 - q^{10n})$$

$$= \prod_{n=1}^{\infty}(1 - q^n)\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q;\ q)_{2n+1}}, \tag{3.10}$$

proves the theorem.

Next, we define the polynomials $E_{n,m}(q)$ by

$$E_{n,m}(q) = \begin{cases} B_{n,m}(q) + B_{n-1,m}(q) & \text{if } 0 \le m \le n - 1, \\ \begin{bmatrix} n + 1 \\ n \end{bmatrix} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

To give a combinatorial interpretation of the polynomials $B_{n,m}(q)$ and $E_{n,m}(q)$, we consider an integer triangle whose entries $e_{n,k}$ ($n = 0, 1, 2, \ldots;\ 0 \le k \le n$) are given by

$$e_{n,k} = b_{n,k} + b_{n-1,k}, \tag{3.12}$$

where $b_{n,k}$ is the $(k + 1)^{\text{th}}$ coefficient in the expansion of $(x + y)^{2n+1-k}$ when $0 \le k \le n$, and $b_{n,k} = 0$ for $k > n$.

It can be shown that

$$\sum_{k=0}^{n} b_{n,k} = F_{2n+2} \quad \text{and} \quad \sum_{k=0}^{n} e_{n,k} = L_{2n+1}.$$

Note that $E_{n,m}(q)$ and $B_{n,m}(q)$ are q-extensions of the numbers $e_{n,m}$ and $b_{n,m}$ respectively. Moreover, $B_{n,m}(1) = b_{n,m}$ is the number of lattice paths from (1, 0) to $(2n + 1 - m, m)$ with no two successive vertical steps. Defining $E_n(q)$ by

$$E_n(q) = \sum_{k=0}^{n}\begin{bmatrix} 2n + 1 \\ n - k \end{bmatrix}\bar{C}_k(q)(-1)^{n-k}, \tag{3.13}$$

it is easy to show that $E_n(q)$ is a polynomial in q where the sum of the coefficients is equal to unity.

Note also that (2.7) and (3.13) are q-analogues of (1.5).

Finally, we set

$$D_n(q) = \sum_{m=0}^{n} B_{n,m}(q), \tag{3.14}$$

and observe that $D_n(q)$ is a q-analogue of W_{n+1}, where W_n is the weighted composition function with weights 1, 2, ..., n [7, p. 39]; hence, (3.5) leads to the formula

$$L_{2n+1} = W_{n+1} + W_n,\ n \geq 1. \tag{3.15}$$

Note that the sum of the coefficients of $D_n(q)$ is the Fibonacci number F_{2n+2}.

We close this section with the following theorem, which is easy to prove.

Theorem 3.2: Let $\overline{C}_\infty(q)$ be defined by (3.7) and $D_\infty(q) = \lim_{n\to\infty} D_n(q)$, then

$$D_\infty(q) = \frac{1}{2}\overline{C}_\infty(q). \tag{3.16}$$

4. Conclusion

We have given several combinatorial interpretations of the polynomials

$C_n(q)$, $D_{n,m}(q)$, $\overline{C}_n(q)$, $B_{n,m}(q)$, and $E_{n,m}(q)$ at $q = 1$,

the most obvious question that arises is: Is it possible to interpret these polynomials as generating functions? We make the following conjectures:

Conjecture 1: In the expansion of $C_n(q)$, the coefficient of q^k $(k \leq 2n - 2)$ equals twice the number of partitions of k into distinct parts.

Conjecture 2: For $1 \leq k \leq n$, let

$A(k, n)$ = the number of partitions of k into parts $\not\equiv 0, \pm 2, \pm 6, \pm 8, 10 \pmod{20}$ + the number of partitions of $k - 1$ into parts $\not\equiv 0, \pm 2, \pm 4, \pm 6, 10 \pmod{20}$.

then the coefficient of q^k in the expansion of $D_n(q)$ equals $A(k, n)$.

Conjecture 3: In the expansion of $\overline{C}_n(q)$, the coefficient of q^k $(k \leq n - 1)$ equals $2A(k, n - 1)$.

Remark: Theorems 2.2, 3.1, and 3.2 are the limiting cases $n \to \infty$ of Conjectures 1, 3, and 2 respectively.

We hope that some interested readers can prove Conjectures 1, 2, and 3.

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