MEASURES OF SETS PARTITIONING BOREL'S SIMPLY NORMAL NUMBERS TO BASE 2 IN [0, 1]

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1. Introduction and Theorem

Let

$$\sum_{i=1}^{\infty} d_i(\omega)2^{-i}, \text{ where } d_i(\omega) = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots,$$

denote the dyadic expansion of any element ω in the closed unit interval [0, 1]. This expansion is unique except when ω is a dyadic rational

$$(2m - 1)2^{-n},\ m = 1, 2, \ldots, 2^{n-1},\ n = 1, 2, \ldots,$$

in which case there are two such expansions, the terminating one concluding with an unending succession of zeros and the nonterminating one concluding with an unending succession of ones. To insure uniqueness, we quite arbitrarily choose the terminating expansion in such a case.

Of particular interest is the asymptotic behavior of

$$p_m(\omega) \equiv m^{-1} \sum_{i=1}^{m} d_i(\omega),$$

the proportion of ones appearing among the first m dyadic places in the expansion of ω, for $m = 1, 2, \ldots$. Borel [2] asserted that "almost all" ω in [0, 1] have the property that the limiting value of this proportion is 1/2. More precisely, if ν is the Lebesgue measure on the class of Borel measurable subsets of [0, 1] and if

$$S \equiv \{\omega : 0 \le \omega \le 1, \lim_{m \to \infty} p_m(\omega) = 1/2\},$$

then $\nu(S) = 1$. Borel's arguments in support of this impressive fact were flawed, but valid proofs were supplied by later workers (see [1]). The set S defines those numbers in [0, 1] which are said to be simply normal to base 2.

The very definition of simply normal numbers induces rather natural families of partitions of [0, 1]. Motivated by the definition of S and the fact that, for each fixed positive real number ε less than 1/2 (to avoid triviality), the inequality

$$|p_m(\omega) - 1/2| > \varepsilon$$

holds for only finitely many values of m for every ω in S, we can sharpen Borel's landmark result by considering the following measurable functions which, moreover, can be defined for all ω in [0, 1]:

$$\ell(\omega, \varepsilon) \equiv \sup\{m : m = 1, 2, \ldots, \text{ and } p_m(\omega) > 1/2 + \varepsilon\}$$

and

$$n(\omega, \varepsilon) \equiv \sum_{m=1}^{\infty} I(\{\omega : 0 \le \omega \le 1, p_m(\omega) > 1/2 + \varepsilon\}),$$

where the supremum of the empty set is 0 and $I(A)$ is the indicator function of the set A. Thus, in the expansion of ω, $\ell(\omega, \varepsilon)$ is the "largest" dyadic place, and $n(\omega, \varepsilon)$ is the total "number" of dyadic places, at which the proportion of ones up to that place exceeds $1/2 + \varepsilon$. Note that these functions assume the value $+\infty$ for infinitely many ω in $[0, 1]$, but Borel's result implies that the sets on which they assume an infinite value have Lebesgue measure zero.

For every ω in S, the values of these functions are nonnegative integers. It is illuminating, therefore, to decompose S according to the values of each of these functions, creating the families of countable partitions $\mathfrak{L}(\varepsilon)$ and $\mathfrak{N}(\varepsilon)$ having respective members

$$L_j \equiv \{\omega : \omega \in S,\ \ell(\omega, \varepsilon) = j\}, \quad j = 0, 1, 2, \ldots,$$

and

$$N_j \equiv \{\omega : \omega \in S,\ n(\omega, \varepsilon) = j\}, \quad j = 0, 1, 2, \ldots .$$

The following theorem gives the Lebesgue measures of the members of each of these partitions when $\varepsilon = k/(2k + 4)$ for any positive integer k.

Theorem: Suppose $\varepsilon = k/(2k + 4)$ for some positive integer k. Then

$$\nu(L_0) = \nu(N_0) = 1 - \gamma_k,$$

and for $j = 1, 2, \ldots,$

$$\nu(L_j) = \left[1 - \gamma_k^{(k+2)([\![j/(k+2)]\!]+1)-j}\right]\binom{j}{[\![j/(k + 2)]\!]}2^{-(j+1)}$$

if $j \neq 0 \bmod (k + 2)$; whereas $\nu(L_j) = 0$ if $j = 0 \bmod (k + 2)$, and

$$\nu(N_j) = (1 - \gamma_k)2^{-j}\sum_{i=0}^{[\![j/(k+2)]\!]}[1 - (k + 2)i/j]\binom{j}{i}.$$

Here, γ_k is the unique solution of $x^{k+2} - 2x + 1 = 0$ in the open interval $(0, 1)$ and $[\![t]\!]$ is the greatest integer not exceeding t.

Remark 1: If $j = r \bmod (k + 2)$, where $r = 0, 1, \ldots, k + 1$, then we have that

$$(k + 2)([\![j/(k + 2)]\!] + 1) - j = k + 2 - r.$$

Remark 2: For $k = 1, 2, 3, 4$, and 5 and $k \to \infty$, the values of $\nu(L_j)$ are tabled in [3] for

$$j = 0, 1, \ldots, \inf\left\{h : \sum_{j=0}^{h}\nu(L_j) \geq 0.9999\right\},$$

and the values of $\nu(N_j)$ are tabled in [7] for

$$j = 0, 1, \ldots, \inf\left\{h : \sum_{j=0}^{h}\nu(N_j) \geq 0.9999\right\}.$$

Remark 3: Our theorem remains true if $p_m(\omega)$ is interpreted as the proportion of zeros appearing among the first m dyadic places in the expansion of ω for $m = 1, 2, \ldots$. Furthermore, since the proportion of zeros exceeds $1/2 + \varepsilon$ if and only if the proportion of ones is less than $1/2 - \varepsilon$, our theorem remains valid when the strict inequalities are reversed and ε is replaced by $-\varepsilon$ in the definitions of $\ell(\omega, \varepsilon)$ and $n(\omega, \varepsilon)$.

Note: Because

$$x^{k+2} - 2x + 1 = (x - 1)\left(\sum_{i=1}^{k+1}x^i - 1\right)$$

and, for $0 < x \leq 1/2$,

$$\sum_{i=1}^{k+1}x^i < 1,$$

γ_k is the unique solution of

$$\sum_{i=1}^{k+1} x^i = 1 \text{ in } (1/2,\ 1) \text{ for every positive integer } k.$$

We now show that $\gamma_k = r_{k+1}^{-1}$, the reciprocal of the $(k+1)^{\text{st}}$ Fibonacci root tabled in [5] for $k = 1, 2, \ldots, 18$. For any positive integer $K \geq 2$, consider the K-generalized Fibonacci numbers defined by $f_K(j) = 0$, for $j = 0, \ldots, K - 2$, $f_K(K - 1) = 1$, and

$$f_K(j) = \sum_{i=1}^{K} f_K(j - i) \text{ for } j = K,\ K + 1,\ \ldots,$$

and tabled in [5] for $K = 2, \ldots, 7$ and $j = 0, \ldots, 15$. Miles [6] proved that

$$\lim_{j\to\infty} f_K(j + 1)/f_K(j) = r_K,$$

where r_K is the unique solution of

$$\sum_{i=0}^{K-1} x^i = x^K \text{ in } (1,\ 2).$$

It follows that r_K^{-1} is the unique solution of

$$\sum_{i=1}^{K} x^i = 1 \text{ in } (1/2,\ 1);$$

hence, $\gamma_k = r_{k+1}^{-1}$ for $k = 1, 2, \ldots$.

2. Proof of the Theorem

If S^c denotes the complement of S with respect to $[0, 1]$, then $\nu(S^c) = 0$, and since, for $j = 0, 1, 2, \ldots,$

$$\{\omega : 0 \leq \omega \leq 1,\ \ell(\omega, \varepsilon) = j\} = L_j \cup \{\omega : \omega \in S^c,\ \ell(\omega, \varepsilon) = j\},$$

it follows that

$$\nu(L_j) = \nu(\{\omega : 0 \leq \omega \leq 1,\ \ell(\omega, \varepsilon) = j\}).$$

Similarly, for every nonnegative integer j,

$$\nu(N_j) = \nu(\{\omega : 0 \leq \omega \leq 1,\ n(\omega, \varepsilon) = j\}).$$

Now it is well known (see, e.g., [4], Ex. 4, p. 56) that $\langle d_i(\omega)\rangle$ is a sequence of independent random variables (functions) on $[0, 1]$ for which

$$p \equiv \nu(\{\omega : 0 \leq \omega \leq 1,\ d_i(\omega) = 1\}) = 1/2$$

and

$$q \equiv \nu(\{\omega : 0 \leq \omega \leq 1,\ d_i(\omega) = 0\}) = 1/2$$

for every positive integer i, since $d_i(\omega) = 1$ on 2^{i-1} disjoint intervals each of length 2^{-i}, and similarly for $d_i(\omega) = 0$. Note that

$$\{\langle d_i(\omega)\rangle : 0 \leq \omega \leq 1\}$$

differs from the set of all sequences of zeros and ones only by the set of sequences corresponding to the nonterminating expansions of the set of dyadic rationals mentioned above. As this latter set is countable and, hence, of measure zero, its inclusion or exclusion has no effect in our work.

If we define the Rademacher functions

$$x_i(\omega) = 2d_i(\omega) - 1,\ i = 1, 2, \ldots,$$

so that $\langle x_i(\omega)\rangle$ is a sequence of independent and identically distributed random variables such that $x_i(\omega) = +1$ or -1 with respective probabilities $p = 1/2$ and $q = 1/2$, then $p_m(\omega) > 1/2 + \varepsilon$ if and only if $s_m(\omega) > 2\varepsilon m$, where

$$s_m(\omega) \equiv \sum_{i=1}^{m} x_i(\omega) \text{ for every positive integer } m.$$

Our theorem then follows immediately from the theorems in [3] and [7], where

$$\mu = p - q = 0 \quad \text{and} \quad \lambda = 2\varepsilon = k/(k + 2),\ k = 1, 2, \ldots .$$

3. The Special Case $\varepsilon = 1/6$

The case in which $\varepsilon = 1/6$ ($k = 1$) is particularly attractive since it is the smallest ε dealt with by our theorem and since γ_1, the unique solution of $x^3 - 2x + 1 = 0$ in $(0, 1)$, is $\phi \equiv (\sqrt{5} - 1)/2$, the reciprocal of the ubiquitous golden ratio. In this case, our theorem yields $\nu(L_0) = 1 - \phi = \phi^2$ and, for $j = 0, 1, \ldots,$

$$\nu(L_{3j+1}) = \phi\binom{3j+1}{j}2^{-3j-2},$$

and

$$\nu(L_{3j+2}) = \phi^2\binom{3j+2}{j}2^{-3j-3} = [\phi(3j + 2)/(4j + 4)]\nu(L_{3j+1}),$$

with $\nu(L_{3j+3}) = 0$. Here, the successive values of $\nu(L_{3j+1})$ are most easily computed recursively using $\nu(L_1) = \phi/4$ and the relation

$$\nu(L_{3j+4}) = \frac{3(3j + 4)(3j + 2)}{16(j + 1)(2j + 3)}\nu(L_{3j+1}), \quad j = 0, 1, 2, \ldots .$$

It follows that, for $j = 0, 1, 2, \ldots,$

$$\nu(L_{3j+1}) > \nu(L_{3j+2}) > \nu(L_{3j+3}) = 0$$

and

$$\nu(L_{3j+1}) > \nu(L_{3j+4})$$

so that, for increasing values of the subscript, these measures exhibit an interesting "damped saw-tooth" pattern, each value of j corresponding to a single tooth.

It is noteworthy to observe that

$$\begin{aligned}\phi = 1 - \nu(L_0) &= 1 - \nu(\{\omega : \omega \in S,\ p_m(\omega) \le 2/3 \ \ \forall m = 1, 2, \ldots\})\\ &= \nu(\{\omega : \omega \in S,\ p_m(\omega) > 2/3 \text{ for some } m = 1, 2, \ldots\}),\end{aligned}$$

that is, the set E of simply normal numbers to base 2 in [0, 1] having the property that the proportion of ones to some dyadic place in their expansion exceeds 2/3 has measure ϕ. Clearly, $S \cap [1/2, 1]$, with measure 1/2, is a subset of E. Yet, E is dense in [0, 1]. For if η is an arbitrarily small but fixed positive real number, then for any

$$\omega = \sum_{i=1}^{\infty} d_i(\omega)2^{-i} \text{ in } [0, 1],$$

consider

$$\omega' = \sum_{i=1}^{N} d_i(\omega)2^{-i} + \sum_{j=1}^{2N+1} 2^{-(N+j)} + \sum_{k=1}^{\infty} 2^{-(3N+2k)},$$

where N is the smallest positive integer such that $2^{-N} < \eta$. Here,

$$p_m(\omega') = m^{-1}\left[\sum_{i=1}^{N} d_i(\omega) + (2N + 1) + [\![(m - 3N)/2]\!]\right], \text{ for } m > 3N + 1,$$

so that $\lim_{m\to\infty} p_m(\omega') = 1/2$; hence, $\omega' \in S$. Moreover,

$$p_{3N+1}(\omega') = (3N + 1)^{-1}\left[\sum_{i=1}^{N} d_i(\omega) + (2N + 1)\right] \ge (2N + 1)/(3N + 1) > 2/3;$$

therefore, $\omega' \in E$. Finally, since ω and ω' agree in the first N dyadic places of their expansions, we have $|\omega' - \omega| \le 2^{-N} < \eta$.

It is also worth noting that the measures of the members of $\mathcal{L}(1/6)$ given above yield a simple formula expressing ϕ in terms of the series

$$y \equiv \sum_{j=0}^{\infty} \binom{3j+1}{j} 2^{-3j} \quad \text{and} \quad z \equiv \sum_{j=0}^{\infty} \binom{3j+2}{j} 2^{-3j}.$$

For,

$$\sum_{j=0}^{\infty} \nu(L_j) = \nu(S) = 1 = \phi^2 + \phi$$

implies $\phi y/4 + \phi^2 z/8 = \phi$; hence, $\phi = 2(4 - y)/z$. Note that

$$y/4 = 1/(\phi\sqrt{5}) \quad \text{and} \quad z/8 = 1/\sqrt{5}.$$

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Announcement

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