SOME SEQUENCES ASSOCIATED WITH THE GOLDEN RATIO*

Tom C. Brown and Allen R. Freedman
Simon Fraser University, Burnaby, B.C., Canada V5A IS6
(Submitted June 1989)

A number of people have considered the arithmetical, combinatorial, geometrical and other properties of sequences of the form $([n\alpha]: n \geq 1)$, where α is a positive irrational number and $[\ ]$ denotes the greatest integer function. (See, e.g., [1]-[16]) and the references contained in those papers, especially [8] and [16].)

There are several other sequences which may be naturally associated with the sequence $([n\alpha]: n \geq 1)$. They are the *difference sequence*

$$f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$$

(the difference sequence is "normalized" by subtracting $[\alpha]$ so that its values are 0 and 1), the *characteristic function*

$$g_\alpha(n) \quad (g_\alpha(n) = 1 \text{ if } n = [k\alpha] \text{ for some } k, \text{ and } g_\alpha(n) = 0 \text{ otherwise}),$$

and the *hit sequence*

$$h_\alpha(n),$$

where $h_\alpha(n)$ is the number of different values of k such that $[k\alpha] = n$.

We use the notation

$$f_\alpha = (f_\alpha(n): n \geq 1),\ g_\alpha = (g_\alpha(n): n \geq 1),\ h_\alpha = (h_\alpha(n): n \geq 1).$$

Note that $f_\alpha = f_{\alpha+k}$ for any integer $k \geq 1$. In particular, $f_\alpha = f_{\alpha-1}$ if $\alpha > 1$.

Special properties of these sequences in the case where α equals τ, the golden mean, $\tau = (1+\sqrt{5})/2$, are considered in [5], [12], [14], and [16]. For example, the following is observed in [12]. Let $u_n = [n\tau]$, $n \geq 1$, and let F_k denote the k^{th} Fibonacci number. Given k, let $r = F_{2k}$, $s = F_{2k+1}$, $t = F_{2k+2}$. Then

$$u_r = s,\ u_{2r} = 2s,\ u_{3r} = 3s,\ \ldots,\ u_{(t-2)r} = (t-2)s;$$

thus, the sequence $([n\tau])$ contains the $(t-2)$-term arithmetic progression $(s, 2s, 3s, \ldots, (t-2)s)$.

It was shown in [16], using a theorem of A. A. Markov [11] (which describes the sequence f_α (for any α) explicitly in terms of the simple continued fraction expansion of α), that the difference sequence f_τ has a certain "substitution property." We give a simple proof of this below (Theorem 2) without using Markov's theorem. We also make several observations concerning the three sequences f_τ, g_τ, and h_τ.

Theorem 1: The golden mean τ is the smallest positive irrational real number α such that $f_\alpha = g_\alpha = h_\alpha$. In fact, $f_\alpha = g_\alpha = h_\alpha$ exactly when $\alpha^2 = k\alpha + 1$, where $k = [\alpha] \geq 1$.

Proof: It follows directly from the definitions (we omit the details) that if α is irrational and $\alpha > 1$, then $h_\alpha = g_\alpha = f_{1/\alpha}$. (The fact that $g_\alpha = f_{1/\alpha}$ is mentioned in [8]. It is straightforward to show that

$$g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1 \quad\text{and}\quad g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0.)$$

*Research partially supported by NSERC.

Also, if α is irrational and $\alpha > 0$, then

$$h_\alpha(n) = f_{1/\alpha}(n) + [1/\alpha] \text{ for all } n \geq 1.$$

Thus, if α is irrational and $f_\alpha = g_\alpha = h_\alpha$, then $\alpha > 1$ (otherwise, g_α is identically equal to 1, and f_α is not) and

$$f_{\alpha - [\alpha]}(n) = f_\alpha(n) = g_\alpha(n) = f_{1/\alpha}(n) \text{ for all } n \geq 1.$$

Since the sequence f_β determines β if $\beta < 1$, this gives $\alpha - [\alpha] = 1/\alpha$, and the result follows.

Definition: For any finite or infinite sequence w consisting of 0's and 1's, let $\overline{w}$ be the sequence obtained from w by replacing each 0 in w by 1, and each 1 by 10. For example, $\overline{10110} = 10110101$. (Compare "Fibonacci strings" [10, p. 85].)

Note that $\overline{uv} = \overline{u} \cdot \overline{v}$, and that $\overline{u} = \overline{v} \Rightarrow u = v$ by induction on the length of v.

Theorem 2: The sequences f_τ and $\overline{f_\tau}$ are identical.

Proof: First, we show that if $0 < \alpha < 1$, then $\overline{f_\alpha} = g_{1+\alpha}$. Let $L(w)$ denote the *length* of the finite sequence w, so that if $w = f_\alpha(1)f_\alpha(2) \ldots f_\alpha(k)$, then

$$L(\overline{w}) = k + f_\alpha(1) + \cdots + f_\alpha(k) = k + [(k+1)\alpha].$$

Thus,

$$\begin{aligned}[\overline{f_\alpha}(n) = 1] &\Leftrightarrow [n = L(\overline{w}) + 1 \text{ for some initial segment } w \text{ of } f_\alpha] \\ &\Leftrightarrow [n = [(k+1)(1+\alpha)] \text{ for some } k \geq 0] \Leftrightarrow [g_{1+\alpha}(n) = 1].\end{aligned}$$

Therefore, $\overline{f_\tau} = \overline{f_{\tau - 1}} = g_\tau = f_{1/\tau} = f_{\tau - 1} = f_\tau$.

Corollary 1: The sequence f_τ can be generated by starting with $w = 1$ and repeatedly replacing w by $\overline{w}$.

Proof: If we define $E_1 = 1$ and $E_{k+1} = \overline{E_k}$, then, since $\overline{1} = 10$ begins with a 1, it follows that, for each k, E_k is an initial segment of E_{k+1}. By Theorem 2 and induction, each E_k is an initial segment of f_τ. Thus,

$$E_1 = 1,\ E_2 = \overline{E_1} = 10,\ E_3 = \overline{E_2} = 101,\ E_4 = \overline{E_3} = 10110,$$
$$E_5 = \overline{E_4} = 10110101, \text{ etc.},$$

are all initial segments of f_τ. (These blocks naturally have lengths 1, 2, 3, 5, 8, ...)

Corollary 2: For each $i \geq 1$, let x_i denote the number of 1's in the sequence f_τ which lie between the i^{th} and $(i+1)^{\text{st}}$ 0's. Thus,

$$f_\tau = 1011010110110101101011011010110 11\ldots,$$
$$(x_n) = \quad 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ \ 2 \ldots .$$

Then the sequences $(x_n - 1)$ and f_τ are identical.

Proof: If we start with the sequence (x_n) and replace each 1 by 10 and each 2 by 101, we obtain the sequence f_τ. Since $\overline{\overline{0}} = 10$ and $\overline{\overline{1}} = 101$, this shows that $(\overline{\overline{x_n - 1}}) = f_\tau = \overline{\overline{f_\tau}}$. Therefore, $(\overline{x_n - 1}) = \overline{f_\tau}$, and, finally, $(x_n - 1) = f_\tau$.

References

1. T. Bang. "On the Sequence $[n\alpha]$." *Math. Scand.* 5 (1957):69-76.
2. I. G. Connell. "A Generalization of Wythoff's Game." *Can. Math. Bull.* 2 (1959):181-90.
3. I. G. Connell. "Some Properties of Beatty Sequences I." *Can. Math. Bull.* 2 (1959):190-97.

4. I. G. Connell. "Some Properties of Beatty Sequences II." *Can. Math. Bull.* *3* (1960):17-22.
5. H. S. M. Coxeter. "The Golden Section, Phyllotzxis, and Wythoff's Game." *Scripta Math.* *19* (1953):135-43.
6. A. S. Fraenkel. "The Bracket Function and Complementary Sets of Integers." *Can. J. Math.* *21* (1969):6-27.
7. A. S. Fraenkel & J. Levitt. "Characterization of the Set of Values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$." *Discrete Math.* *2* (1972):335-45.
8. A. S. Fraenkel, M. Mushkin, & U. Tassa. "Determination of $[n\theta]$ by Its Sequence of Differences." *Canad. Math. Bull.* *21* (1978):441-46.
9. R. L. Graham. "Covering the Positive Integers by Disjoint Sets of the Form $\{[n\alpha + \beta]: n = 1, 2, \ldots\}$." *J. Combinatorial Th.* *(A) 15* (1973):354-58.
10. D. Knuth. *The Art of Computer Programming*. Vol. I. 2nd ed. New York: Addison-Wesley, 1973.
11. A. A. Markoff. "Sur une question de Jean Bernoulli." *Math. Ann.* *19* (1882): 27-36.
12. N. S. Mendelsohn. "The Golden Ratio and van der Waerden's Theorem." *Proc. 5th SE Conf. on Combinatorics, Graph Theory, and Computing, Congressus Numerantium* *10* (1974):93-109.
13. I. Niven. *Diophantine Approximations*. New York: Wiley, 1963.
14. J. Rosenblatt. "The Sequence of Greatest Integers of an Arithmetic Progression." *J. London Math. Soc.* *(2) 17* (1978):213-18.
15. Th. Skolem. "On Certain Distributions of Integers in Pairs with Given Differences." *Math. Scand.* *5* (1957):57-68.
16. K. B. Stolarsky. "Beatty Sequences, Continued Fractions, and Certain Shift Operators." *Canad. Math. Bull.* *19* (1976):473-82.

*****