ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-688 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Find the number of increasing sequences of integers such that 1 is the first term, n is the last term, and the difference between successive terms is 1 or 2. [For example, if $n = 8$, then one such sequence is 1, 2, 3, 5, 6, 8 and another is 1, 3, 4, 6, 8.]

B-689 *Proposed by Philip L. Mana, Albuquerque, NM*

Show that $F_n^2 - 1$ is a sum of Fibonacci numbers with distinct positive even subscripts for all integers $n \geq 3$.

B-690 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k = \alpha^{10k+1} + \alpha^{10k+2} + \alpha^{10k+3} + \cdots + \alpha^{10k+10}$, where $\alpha = (1 + \sqrt{5})/2$. Find positive integers b and c such that $S_k/\alpha^{10k+b} = c$ for all nonnegative integers k.

B-691 *Proposed by Heiko Harborth, Technische Universität Braunschweig, West Germany*

Herta T. Freitag asked whether a golden rectangle can be inscribed into a larger golden rectangle (all four vertices of the smaller are points on the sides of the larger one). An answer follows from the solution of the generalized problem: Which rectangles can be inscribed into larger similar rectangles?

B-692 *Proposed by Gregory Wulczyn, Lewisburg, PA*

Let $G(a, b, c) = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}$. Prove or disprove that each of F_{a+b+c}, F_{b+c-a}, F_{c+a-b}, and F_{a+b-c} is an integral divisor of $G(a, b, c)$ for all odd positive integers a, b, and c.

B-693 *Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Tech, Atlanta, GA*

Let A consist of all pairs $\{x, y\}$ chosen from $\{1, 2, \ldots, 2n\}$, B consist of all pairs from $\{1, 2, \ldots, n\}$, and C of all pairs from $\{n + 1, n + 2, \ldots, 2n\}$. Let S consist of all sets $T = \{P_1, P_2, \ldots, P_k\}$ with the P_i (distinct) pairs in A. How many of the T in S satisfy at least one the the conditions:

(i) $P_i \cap P_j \neq \emptyset$ for some i and j, with $i \neq j$,

(ii) $P_i \in B$ for some i, or

(iii) $P_i \in C$ for some i?

SOLUTIONS

Limit of Nested Square Roots

B-664 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Let $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for n in $\{0, 1, \ldots\}$. Show that

$$\lim_{n\to\infty} a_n = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\binom{i}{j}\right]^{-1}.$$

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

First

$$\sum_{0\le j\le i}\binom{i}{j} = 2^i \quad \text{and} \quad \sum_{0\le i\le\infty} 2^{-i} = 2.$$

Next,

$$a_{n+1}^2 > a_n^2 \text{ iff } a_n + 2 > a_{n-1} + 2,$$

and similarly,

$$a_{n+1} < 2 \text{ iff } a_{n+1}^2 = a_n + 2 < 4,$$

implying $a_n < 2$. Thus, an induction shows that the sequence $\{a_n\}$ is monotonely increasing and bounded; hence, the limit, L, exists. Squaring the defining recursion and taking limits we find $L^2 = L + 2$ or $L = 2$.

This solves the problem, since both sides of the problem equation have a value of 2.

Also solved by R. André-Jeannin, Paul S. Bruckman, Pat Costello, Piero Filipponi & Adina DiPorto, C. Georghiou, Norbert Jensen & Uwe Pettke, Hans Kappus, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.

Unique Real Solutions of Cubics

B-665 *Proposed by Christopher C. Street, Morris Plains, NJ*

Show that $AB = 9$, where

$$A = (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3} + 1,$$
$$B = (17 + 3\sqrt{33})^{1/3} + (17 - 3\sqrt{33})^{1/3} - 1.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

Put

$$(19 + 3\sqrt{33})^{1/3} = a, \quad (19 - 3\sqrt{33})^{1/3} = a'.$$

Then $a^3 + (a')^3 = 38$, $aa' = 4$, and we have

$$(A - 1)^3 = (a + a')^3 = 3aa'(a + a') + a^3 + (a')^3 = 12(A - 1) + 38.$$

Therefore, $f(A) = 0$, where

$$f(x) = x^3 - 3x^2 - 9x - 27.$$

In the same way, we find that $g(B) = 0$, where

$$g(x) = x^3 + 3x^2 + 9x - 27.$$

It is easily seen that the polynomials f and g have exactly one real zero each, which must therefore be A and B, respectively. On the other hand,

$$x^3 f(9/x) = -27g(x);$$

hence, $f(9/B) = 0$, and therefore $A = 9/B$.

Also solved by Paul S. Bruckman, C. Georghiou, Norbert Jensen & Uwe Pettke, L. Kuipers, Y. H. Harris Kwong, Carl Libis, and the proposer.

Diagonal p of Pascal Triangle Modulo p

B-666 *Taken from solutions to B-643 by Russell Jay Hendel, Dowling College, Oakdale, NY, and by Lawrence Somer, Washington, D.C.*

For primes p, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $[x]$ is the greatest integer in x.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

The result follows easily from the same formula of E. Lucas used in the solution to B-643 [vol. 28 (1990), p. 185]. Alternately, let t be the integer satisfying $0 \le t \le p - 1$ and $p \mid (n - t)$. Then $(n - t)/p = [n/p]$ and

$$n(n - 1) \cdots (n - t + 1)(n - t - 1) \cdots (n - p + 1) \equiv (p - 1)! \pmod{p}.$$

Therefore, in the field Z_p of the integers modulo p,

$$\binom{n}{p} = [n/p]\frac{n \cdots (n - t + 1)(n - t - 1) \cdots (n - p + 1)}{(p - 1)!} = \left[\frac{n}{p}\right]$$

Hence, $\binom{n}{p} \equiv [n/p] \pmod{p}$.

Also solved by R. André-Jeannin, Paul S. Bruckman, C. Georghiou, Norbert Jensen & Uwe Pettke, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.

Cyclic Permutation of Digits

B-667 *Proposed by Herta T. Freitag, Roanoke, VA*

Let p be a prime, $p \neq 2$, $p \neq 5$, and m be the least positive integer such that $10^m \equiv 1 \pmod{p}$. Prove that each m-digit (integral) multiple of p remains a multiple of p when its digits are permuted cyclically.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Suppose

$$n = a_{m-1}10^{m-1} + \cdots + a_1 10 + a_0 \equiv 0 \pmod{p}.$$

Let n_t be the integer obtained from n by permuting its digits cyclically by t positions. More specifically,

$$n_t = a_{m-t-1}10^{m-1} + \cdots + a_0 10^t + a_{m-1}10^{t-1} + \cdots + a_{m-t},$$

where $0 \le t \le m-1$. Since $10^m \equiv 1 \pmod{p}$, we have

$$\begin{aligned} n_t &\equiv a_{m-t-1}10^{m-1} + \cdots + a_0 10^t + a_{m-1}10^{m+t-1} + \cdots + a_{m-t}10^m \\ &= 10^t(a_{m-t-1}10^{m-t-1} + \cdots + a_0 + a_{m-1}10^{m-1} + \cdots + a_{m-t}10^{m-t}) \\ &= 10^t n \equiv 0 \pmod{p}. \end{aligned}$$

Also solved by R. André-Jeannin, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen & Uwe Pettke, L. Kuipers, Bob Prielipp, Sahib Singh, Lawrence Somer, and the proposer.

Base 9 Modular Arithmetic Progression

B-668 *Proposed by A. P. Hillman in memory of Gloria C. Padilla*

Let h be the positive integer whose base 9 numeral

100101102...887888

is obtained by placing all the 3-digit base 9 numerals end-to-end as indicated.

(a) What is the remainder when h is divided by the base 9 integer 14?

(b) What is the remainder when h is divided by the base 9 integer 81?

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that

$$\begin{aligned} (100101102\ldots887888)_9 &= (888)_9 9^0 + (887)_9 9^3 + \cdots + (100)_9 9^{3\cdot 647} \\ &= (729-1)9^0 + (729-2)9^3 + \cdots + (729-648)9^{3\cdot 647} \\ &= \sum_{n=1}^{648} (9^3 - n)9^{3n-3}. \end{aligned}$$

(a) Now, since $(14)_9 = 13$ and $9^3 \equiv 1 \pmod{13}$, we have

$$h = \sum_{n=1}^{648}(9^3 - n)9^{3n-3} \equiv \sum_{n=1}^{648}(1 - n) = \frac{647 \cdot 648}{2} \equiv -3 \pmod{13}.$$

(b) Now we have $(81)_9 = 73$ and $9^3 \equiv -1 \pmod{73}$. Therefore,

$$h = \sum_{n=1}^{648}(9^3 - n)9^{3n-3} \equiv -2 + \sum_{n=2}^{648}(-1 - n)(-1)^{n-1} = -1 - \sum_{n=1}^{649} n(-1)^n$$

$$= -1 - \frac{(-1)^{649}(2 \cdot 649 + 1) - 1}{4} = 324 \equiv 32 \pmod{73}.$$

Also solved by Charles Ashbacher, Paul S. Bruckman, and the proposer.

Fibonacci and Lucas Identities

B-669 *Proposed by Gregory Wulczyn, Lewisberg, PA*

Do the equations

$$25F_{a+b+c}F_{a+b-c}F_{b+c-a}F_{c+a-b} = 4 - L_{2a}^2 - L_{2b}^2 - L_{2c}^2 + L_{2a}L_{2b}L_{2c}$$

$$L_{a+b+c}L_{a+b-c}L_{b+c-a}L_{c+a-b} = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}$$

hold for all even integers a, b, and c?

Solution by C. Georghiou, University of Patras, Greece

The answer is "Yes"! From the identity

$$5F_{m+n}F_{m-n} = L_{2m} - (-1)^{m+n}L_{2n}$$

we get [setting $(-1)^{a+b+c} = \mathrm{e}$)

$$\begin{aligned}25F_{a+b+c}F_{a+b-c}F_{a-b+c}F_{c+b-a} &= [L_{2a+2b} - \mathrm{e}L_{2c}][L_{2c} - \mathrm{e}L_{2a-2b}]\\ &= L_{2c}[L_{2a+2b} + L_{2a-2b}] - \mathrm{e}L_{2c}^2 - \mathrm{e}L_{2a+2b}L_{2a-2b}\\ &= L_{2c}[L_{2a}L_{2b}] - \mathrm{e}L_{2c}^2 - \mathrm{e}[L_{4a} + L_{4b}]\\ &= L_{2a}L_{2b}L_{2c} - \mathrm{e}[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4],\end{aligned}$$

and for a, b, and c even (actually for $a + b + c$ even), the given identity is established.

In a similar way, using the identity

$$L_{m+n}L_{m-n} = L_{2m} + (-1)^{m+n}L_{2n}$$

we find

$$\begin{aligned}L_{a+b+c}L_{a+b-c}L_{a-b+c}L_{c+b-a} &= [L_{2a+2b} + \mathrm{e}L_{2c}][L_{2c} + \mathrm{e}L_{2a-2b}]\\ &= L_{2c}[L_{2a+2b} + L_{2a-2b}] + \mathrm{e}L_{2c}^2 + \mathrm{e}L_{2a+2b}L_{2a-2b}\\ &= L_{2a}L_{2b}L_{2c} + \mathrm{e}[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4],\end{aligned}$$

which establishes the second identity.

Also solved by R. André-Jeannin, Paul S. Bruckman, Herta T. Freitag, Norbert Jensen & Uwe Pettke, L. Kuipers, Bob Prielipp, and the proposer.

*****