SECOND-ORDER RECURRENCE AND ITERATES OF $[\alpha n + 1/2]$

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The equation

$$[x[nx + 1/2] + 1/2] = [nx + 1/2] + n$$

determines a unique real number x, in the sense that there is only one value of x for which this equation holds for all positive integers n. This special value of x is the golden mean, $(1 + \sqrt{5})/2$.

The purpose of this note is to prove the above assertion in the more general form of Theorem 1 (of which it is the case when $a = b = 1$), and to give a necessary and sufficient condition that iterates of the function $[\alpha n + 1/2]$, in the sense of Theorem 2, form a second-order recurrence sequence.

Notation: Throughout, let $f(x) = x^2 - ax - b$, where a and b are nonzero integers satisfying $a^2 + 4b > 0$. Write the roots of $f(x)$ as

$$\alpha = (a + \sqrt{a^2 + 4b})/2 \quad \text{and} \quad \beta = a - \alpha.$$

Let $[a_0, a_1, a_2, \ldots]$ denote the continued fraction of the root α, with convergents p_k/q_k given in the usual way (e.g., Roberts [1], pp. 97-100) by

$$p_{-2} = 0,\ p_{-1} = 1,\ p_k = a_k p_{k-1} + p_{k-2} \text{ for } k \ge 0,$$
$$q_{-2} = 1,\ q_{-1} = 0,\ q_0 = 1,\ q_k = a_k q_{k-1} + q_{k-2} \text{ for } k \ge 1.$$

Lemma 1: $|\beta| < 1$ if and only if $|b - 1| < |a|$, and
$|\beta| = 1$ if and only if $|b - 1| = |a|$.

Proof: $|\beta| \le 1$ if and only if

(1) $\quad a - 2 \le \sqrt{a^2 + 4b} \le a + 2,$

with equality if and only if $|\beta| = 1$. This inequality shows that a cannot be less than or equal to -2, since $a^2 + 4b$ is positive. Moreover, if $a = -1$, then $b \ge 1$, so that $\sqrt{a^2 + 4b} > a + 2$, a contradiction. Therefore, $a \ge 1$. In case $a = 1$, we have

(2) $\quad 2 - a \le \sqrt{a^2 + 4b} \le a + 2,$

and if $a \ge 2$, then the leftmost member of inequality (1) is nonnegative. So, if $a = 1$, square the members of inequality (2), and if $a \ge 2$, square those of inequality (1). In both cases, the resulting inequalities easily simplify to $-a \le b - 1 \le a$.

Lemma 2: There exists a positive integer K such that

$$[(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2] < bq_k < [(p_k - aq_k + 1)(p_k + 1)/q_k + 1/2]$$

for all $k \ge K$.

Proof: It suffices to prove for all large enough k the inequalities

$$(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2 < bq_k + 1$$

and

$$bq_k < (p_k - aq_k + 1)(p_k + 1)/q_k - 1/2,$$

which are equivalent to

(3) $\quad [(p_k - 1)/q_k]^2 + a/q_k < ap_k/q_k + b + 1/2q_k < [(p_k + 1)/q_k]^2 - a/q_k.$

Substitute $\alpha + \varepsilon$ for p_k/q_k, where $|\varepsilon| < 1/q_k q_{k+1}$ (e.g., Roberts [1], p. 100), square where indicated, and use the fact that $\alpha^2 - a\alpha - b = 0$ to see that (3) is equivalent to

$$|\varepsilon q_k(a - 2\alpha - \varepsilon) + 1/2 - 1/q_k| < |a - 2\alpha - 2\varepsilon|,$$

which holds for all large enough k, since, as $k \to \infty$, the left member approaches 1/2, while the right approaches $|a - 2\alpha| = \sqrt{a^2 + 4b} \geq 1$.

Lemma 3: If $|b - 1| < |a|$, then equation (4) below holds for $x = \alpha$ and for all $n \geq 1$.

Proof: By Lemma 1, $|\beta| < 1$, so that the fractional part $r = n\alpha + 1/2 - [n\alpha + 1/2]$ satisfies $|r - 1/2| < 1/2|\beta|$. Since $\beta = a - \alpha$, we have $-1 < (\alpha - a)(1 - 2r) < 1$, so that $0 < (\alpha - a + 1)/2 + (a - \alpha)r < 1$. Since $\alpha^2 = a\alpha + b$, we then have

$$0 < (\alpha - a)(n\alpha + 1/2 - r) + 1/2 - bn < 1,$$

or

$$bn < (\alpha - a)[n\alpha + 1/2] + 1/2 < 1 + bn,$$

so that equation (4) holds for $x = \alpha$.

Theorem 1: Suppose a and b are integers satisfying $|b - 1| < |a|$. Then there exists one and only one number x for which

(4) $\quad [x[nx + 1/2] + 1/2] = a[nx + 1/2] + bn$

for all $n \geq 1$. Explicitly, $x = \alpha = (a + \sqrt{a^2 + 4b})/2$.

Proof: Let n be the denominator q_k of the k^{th} convergent p_k/q_k to the root α of $x^2 - ax - b$. We shall show that in order for (4) to hold for this choice of n, the number x must lie inside infinitely many intervals (L_k, R_k), where

$$L_k = (p_k - 1)/q_k \quad \text{and} \quad R_k = (p_k + 1)/q_k.$$

To see that $x \geq L_k$ for all large enough k, observe that, for $x < L_k$ and all large enough k, we have

$$[x[nx + 1/2] + 1/2 - a[nx + 1/2]] = [(x - a)[nx + 1/2] + 1/2]$$
$$\leq [((p_k - 1)/q_k - a)[p_k - 1 + 1/2] + 1/2]$$
$$\leq [(p_k - aq_k - 1)(p_k - 1)/q_k + 1/2] < bq_k,$$

by Lemma 2. This contradiction to (4) shows that $x \geq L_k$ for all large enough k. Similarly, Lemma 2 shows that $x \leq R_k$ for all large k. It follows that the only viable candidate for x is α, since only this number lies inside infinitely many of the intervals (L_k, R_k).

Lemma 3 shows that the root α does indeed satisfy (4) for all $n \geq 1$.

Theorem 2: For any positive integer n, the sequence $\{s_k\}$ given by

$$s_1 = n,\ s_2 = [\alpha n + 1/2],\ s_3 = [\alpha s_2 + 1/2],\ \ldots,\ s_k = [\alpha s_{k-1} + 1/2]$$

satisfies the recurrence relation $s_k = as_{k-1} + bs_{k-2}$ for all $n \geq 1$ and for all $k \geq 2$ if and only if $|b - 1| \leq |a|$.

Proof; If $|b - 1| < |a|$, then $s_3 = as_2 + bs_1$, according to (4). In fact, for any $k \geq 3$, substituting s_{k-2} for n into (4) yields $[\alpha s_k + 1/2] = as_{k-1} + bs_{k-2}$, as asserted.

Now, if $b - 1 = a$, then $\alpha = a + 1$, so that

$$s_2 = (a + 1)n \quad \text{and} \quad s_3 = (a + 1)s_2 = (a + 1)^2 n = as_2 + bs_1.$$

By induction,

$$s_k = (a + 1)^{k-1}n = as_{k-1} + bs_{k-2} \text{ for all } k \geq 3.$$

Similarly, if $b - 1 = -a$, then

$$s_k = (a - 1)^{k-1}n = as_{k-1} + bs_{k-2} \text{ for all } k \geq 3.$$

If $|b - 1| > |a|$, then $|\beta| > 1$ by Lemma 1. Then the well-known representation $a_1\alpha^m + b_1\beta^m$ for the m^{th} term of any recurrence sequence

$$t_m = at_{m-1} + bt_{m-2},$$

for which $a^2 + 4b > 0$, shows that the sequence

$$\alpha t_m - t_{m+1} = b_1\beta^m(\alpha - \beta)$$

diverges, so that the relation $t_{m+1} = [\alpha t_m + 1/2]$ cannot hold for all m.

In conclusion, we note that the well-known representation

$$F_n = [\alpha F_{n-1} + 1/2]$$

for the n^{th} Fibonacci number in terms of the golden mean, α, *and only one preceding term*, follows from Theorem 2 when $a = b = 1$. Theorem 2 reveals many other second-order recurrence sequences which lend themselves to this sort of first-order recurrence.

Reference

1. Joe Roberts. *Elementary Number Theory*. Cambridge, Mass.: MIT Press, 1977.

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