ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Notice to proposers: To ensure that no submissions have been misfiled by the new editor, all proposers have been notified about the status of their problems that are still on file. If you have submitted a problem for the Elementary Problem section and have not received notification regarding its status, please contact Dr. Rabinowitz.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-700 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that for positive integers m and n,

$$\alpha^m(\alpha L_n + L_{n-1}) = \alpha^n(\alpha L_m + L_{m-1}).$$

B-701 *Proposed by Herta T. Freitag, Roanoke, VA*

In triangles ABC and DEF, $AC = DF = 5F_{2n}$, $BC = L_{n+2}L_{n-1}$, $EF = L_{n+1}L_{n-2}$, and $AB = DE = 5F_{2n+1} - L_{2n+1} + (-1)^{n-1}$. Prove that $\angle ACB = \angle DFE$.

B-702 *Proposed by L. Kuipers, Sierre, Switzerland*

For n a positive integer, let

$$x_n = F_n + \cfrac{1}{L_n + \cfrac{1}{F_n + \cfrac{1}{L_n + \cfrac{1}{\ddots}}}} \quad \text{and} \quad y_n = F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{\ddots}}}}.$$

(a) Find closed form expressions for x_n and y_n.
(b) Prove that $x_n < y_n$ when $n > 1$.

B-703 *Proposed by H.-J. Seiffert, Berlin, Germany*

Prove that for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} F_{2^k}^4 = \frac{F_{2^{n+1}}^2 - 4^n}{5}.$$

B-704 *Proposed by Paul S. Bruckman, Edmonds, WA*

Let a and b be fixed integers. Show that if three integers are of the form $ax^2 + by^2$ for some integers x and y, then their product is also of this form.

B-705 *Proposed by H.-J. Seiffert, Berlin, Germany*

(a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2\binom{2n}{n}} = \frac{\pi^2}{5}$.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{F_{2n}}{n^2\binom{2n}{n}}$.

SOLUTIONS
edited by **A. P. Hillman**

Triangular Divisibility

B-676 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the n^{th} triangular number $n(n+1)/2$. Characterize the positive integers n such that

$$T_n \,\Big|\, \sum_{i=1}^{n} T_i.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

It is immediate that

$$\sum_{i=1}^{n} T_i = (n+2)T_n/3.$$

Therefore, T_n divides $\sum_{i=1}^{n} T_i$ if and only if $n \equiv 1 \pmod 3$.

Also solved by R. André-Jeannin, Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Guo-Gang Gao, Russell Jay Hendel, Joseph J. Kostal, L. Kuipers, Carl Libis, Graham Lord, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Paul Smith, Lawrence Somer, W. R. Utz, and the proposer.

More Triangular Divisibility

B-677 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T_n = n(n+1)/2$. Characterize the positive integers n with

$$\sum_{i=1}^{n} T_i \,\Big|\, \sum_{i=1}^{n} T_i^2.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

A straightforward calculation shows that

$$\sum_{i=1}^{n} T_i^2 = \frac{3n^2 + 6n + 1}{10} \cdot \frac{n+2}{3} T_n = \frac{3n^2 + 6n + 1}{10} \sum_{i=1}^{n} T_i,$$

by the result of B-676. Working mod 10, we see that $3n^2 + 6n + 1$ is a multiple of 10 if and only if

$$n \equiv 1 \pmod{10} \quad \text{or} \quad n \equiv 7 \pmod{10}.$$

Also solved by R. André-Jeannin, Charles Ashbacher, Paul S. Bruckman, Russell Euler, Joseph J. Kostal, L. Kuipers, Carl Libis, Graham Lord, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, and the proposer.

Nontriangular Numbers

B-678 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Show that L_{4n} and L_{4n+3} are never triangular numbers.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall use the following known results in our solution:

(1) $L_{4n} - 2 = 5F_{2n}^2$ for each positive integer n;

(2) $L_{4n+2} + 2 = 5F_{2n+1}^2$ for each nonnegative integer n.

Note: (1) is (I_{16}) and (2) is (I_{17}) on p. 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton Mifflin, 1969).

As immediate corollaries, we have:

(1′) $L_{4n} \equiv 2 \pmod 5$;

(2′) $L_{4n+2} \equiv 3 \pmod 5$.

Next, we establish the following results.

Lemma 1: The sequence of triangular numbers T_n is periodic modulo 5 with a period of 5.

Proof: It suffices to show that $T_{n+5} \equiv T_n \pmod 5$ where n is an arbitrary positive integer.

$$T_{n+5} - T_n = \frac{(n+5)(n+6)}{2} - \frac{n(n+1)}{2} = \frac{(n^2 + 11n + 30) - (n^2 + n)}{2}$$
$$= 5n + 15 \equiv 0 \pmod 5.$$

Lemma 2: Let n be a positive integer. Then T_n is congruent to 0, 1, or 3 modulo 5.

Proof: The claimed result follows from Lemma 1 and the table given below.

n	1	2	3	4	5
T_n	1	3	6	10	15
T_n (mod 5)	1	3	1	0	0

The fact that L_{4n} is never a triangular number follows from (1′) and Lemma 2.

Since, from (1′) and (2′),

$$L_{4n+3} = 2L_{4n+2} - L_{4n},\ L_{4n+3} \equiv 2(3) - 2 \pmod 5,$$

we have

$$L_{4n+3} \equiv 4 \pmod 5.$$

Thus, L_{4n+3} is never a triangular number by Lemma 2.

Also solved by Paul S. Bruckman, H.-J. Seiffert, Sahib Singh, and the proposers.

Product of 4 Lucas Numbers

B-679 *Proposed by R. André-Jeannin, Sfax, Tunisia*

Express $L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ as a polynomial in L_n.

Solution by Guo-Gang Gao, Université de Montréal, Montréal, Canada

It is easy to prove that $L_{2n} = L_n^2 - (-1)^n 2$. Then

$$\begin{aligned} L_{n-2}L_{n+2} &= (\alpha^{n-2} + \beta^{n-2})(\alpha^{n+2} + \beta^{n+2}) \\ &= L_{2n} + (-1)^{n-2}L_4 \\ &= L_n^2 + (-1)^n 5. \end{aligned}$$

Similarly,

$$L_{n-1}L_{n+1} = L_n^2 - (-1)^n 5.$$

Therefore,

$$L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25.$$

Also solved by Paul S. Bruckman, Russell Euler, Herta T. Freitag, Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, Lawrence Somer, and the proposer.

Congruence

B-681 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let n be a nonnegative integer, $k \geq 2$ an even integer, and $r \in \{0, 1, \ldots, k - 1\}$. Show that

$$F_{kn+r} \equiv (F_{k+r} - F_r)n + F_r \pmod{L_k - 2}.$$

Solution by Guo-Gang Gao, Université de Montréal, Montréal, Canada

Let us first prove that

$$F_{k(n+1)+r} = F_{kn+r}L_k - F_{k(n-1)+r},$$

where $k \geq 2$ is an even integer and $r \geq 0$. Notice that

$$(\alpha \times \beta)^k = (-1)^k = 1.$$

$$F_{kn+r}L_k = \frac{1}{\sqrt{5}}(\alpha^{kn+r} - \beta^{kn+r})(\alpha^k + \beta^k)$$

$$= \frac{1}{\sqrt{5}}(\alpha^{k(n+1)+r} - \beta^{k(n+1)+r}) + \frac{1}{\sqrt{5}}(\alpha^{k(n-1)+r} - \beta^{k(n-1)+r})$$

$$= F_{k(n+1)+r} + F_{k(n-1)+r}.$$

Use mathematical induction for the proof:

(1) It is trivially true when $n = 0, 1$.

(2) Assume that the claim holds for up to n.

Then, by the inductive hypothesis, we have the following:

$$\begin{aligned} F_{k(n+1)+r} &= F_{kn+r}L_k - F_{k(n-1)+r} \\ &\equiv ((F_{k+r} - F_r)n + F_r)L_k \\ &\qquad - ((F_{k+r} - F_r)(n-1) + F_r) \pmod{L_k - 2} \\ &\equiv 2((F_{k+r} - F_r)n + F_r) \\ &\qquad - ((F_{k+r} - F_r)(n-1) + F_r) \pmod{L_k - 2} \\ &\equiv (F_{k+r} - F_r)(n+1) + F_r \pmod{L_k - 2}. \end{aligned}$$

This completes the proof.

Also solved by Paul S. Bruckman, Bob Prielipp, and the proposer.

Lucas Triangular Numbers

B-682 *Proposed by Joseph J. Kostal, University of Illinois, Chicago, IL*

Let $T(n)$ be the triangular number $n(n+1)/2$. Show that

$$T(L_{2n}) - 1 = \frac{1}{2}(L_{4n} + L_{2n}).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

We have

$$T(L_{2n}) - 1 = (L_{2n}^2 + L_{2n} - 2)/2 = (L_{4n} + L_{2n})/2,$$

since it is well known that $L_{2n}^2 - 2 = L_{4n}$.

Also solved by Charles Ashbacher, Scott H. Brown, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, H.-J. Seiffert, Mohammad Parvez Shaikh, Sahib Singh, Lawrence Somer, and the proposer.

LT-Composite

B-683 *Proposed by Joseph J. Kostal, University of Illinois, Chicago, IL*

Let $L(n) = L_n$ and $T_n = n(n+1)/2$. Show that

$$L(T_{2n}) = L(2n^2)L(n) + (-1)^{n+1}L(2n^2 - n).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

We have $L(T_{2n}) = L(2n^2 + n)$. But

$$\begin{aligned} L(2n^2 + n) - L(2n^2)L(n) &= \alpha^{2n^2+n} + \beta^{2n^2+n} - \alpha^{2n^2+n} - \beta^{2n^2+n} \\ &\qquad - \alpha^{2n^2}\beta^n - \alpha^n\beta^{2n^2} \\ &= -(\alpha\beta)^n[\alpha^{2n^2-n} + \beta^{2n^2-n}] \\ &= (-1)^{n+1}L(2n^2 - n), \end{aligned}$$

which proves the assertion.

Also solved by Charles Ashbacher, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

B-680 ***Will be published in the next issue as an error was detected just before publication.***

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