ZECKENDORF REPRESENTATIONS USING NEGATIVE FIBONACCI NUMBERS

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(Submitted May 1990)

It is well known that every positive integer can be represented uniquely as a sum of distinct, nonconsecutive Fibonacci numbers (see, e.g., Brown [1]). This representation is called the Zeckendorf representation of the positive integer. Other Zeckendorf-type representations where the Fibonacci numbers are not necessarily consecutive are possible. Brown [2] considers one where a maximal number of distinct Fibonacci numbers are used rather than a minimal number.

We show here that every integer can be represented uniquely as a sum of nonconsecutive Fibonacci numbers F_i where $i \le 0$ and we specify an algorithm that leads to this representation. We also show that no maximal representation of this form is possible.

We note that for all integers i,

$$F_{-i} = (-1)^{i+1}F_i$$

and

(1) $$F_{i+1} = F_i + F_{i-1}.$$

We note further that $F_0 = 0$, F_{-1}, F_{-3}, F_{-5}, ... are positive and F_{-2}, F_{-4}, ... are negative. Also for $i > 1$,

$$|F_{-i}| < |F_{-i-1}|.$$

The four lemmas below will show that the algorithm that follows them is effective.

Lemma 1: If $n, k > 0$ and $-F_{-2k} \le n < F_{-2k-1} - 1$ then, for some ℓ, $k > \ell > 0$,

$$-F_{-2k+2\ell-1} \le n - F_{-2k-1} < -F_{-2k+2\ell+1} < 0.$$

If $n = F_{-2k-1} - 1$, then

$$n - F_{-2k-1} = -F_{-1}.$$

Proof: If $-F_{-2k} \le n < F_{-2k-1} - 1$, then

$$1 < F_{-2k-1} - n \le F_{-2k-1} + F_{-2k},$$

i.e.,

$$1 < F_{-2k-1} - n \le F_{-2k+1} = F_{2k-1}.$$

Now every integer $p > 1$ is in a range $0 < F_{2m-3} < p \le F_{2m-1}$ where $m \ge 2$.

We must, if $p = F_{-2k-1} - n$, then have $m + \ell = k + 1$ for some $\ell > 0$ and so:

$$0 < F_{2k-2\ell-1} < F_{-2k-1} - n \le F_{2k-2\ell+1};$$

thus,

$$-F_{-2k+2\ell-1} \le n - F_{-2k-1} < -F_{-2k+2\ell+1} < 0.$$

Lemma 2: If $n, k > 0$ and $F_{-2k+1} < n \le -F_{-2k}$ then, for some ℓ, $k > \ell > 0$:

$$0 \le -F_{-2k+2\ell+2} < n - F_{-2k+1} \le -F_{-2k+2\ell}.$$

Proof: If $F_{-2k+1} < n \le -F_{-2k}$, then

$$0 < n - F_{-2k+1} \le -F_{-2k} - F_{-2k+1},$$

so

$$0 < n - F_{-2k+1} \le -F_{-2k+2} = F_{2k-2}.$$

Now every positive integer p is in the range

$$0 \le F_{2m-4} < p \le F_{2m-2}$$

where $m \ge 2$.

We must, if $p = n - F_{-2k+1}$, then have $m + \ell = k + 1$ for some ℓ, $k > \ell > 0$, and so

$$0 \le F_{2k-2\ell-2} < n - F_{-2k+1} \le F_{2k-2\ell},$$

i.e.,

$$0 \le -F_{-2k+2\ell+2} < n - F_{-2k+1} \le -F_{-2k+2\ell}.$$

Lemma 3: If $n < 0$, $k > 0$, and $1 + F_{-2k} < n \le -F_{-2k+1}$ then, for some ℓ, $k > \ell > 0$,

$$0 \le -F_{-2k+2\ell+2} < n - F_{-2k} \le -F_{-2k+2\ell}.$$

If $n = F_{-2k} + 1$,

$$n - F_{-2k} = F_{-1}.$$

Proof: If $1 + F_{-2k} < n \le -F_{-2k+1}$, then

$$1 < n - F_{-2k} \le -F_{-2k+2} = F_{2k-2}$$

and as in the proof of Lemma 2,

$$0 \le F_{2k-2\ell-2} < n - F_{-2k} \le F_{2k-2\ell} \text{ for some } \ell,\ k > \ell > 0;$$

thus,

$$0 \le -F_{-2k+2\ell+2} < n - F_{-2k} \le -F_{-2k+2\ell}.$$

Lemma 4: If $n < 0$, $k > 0$, and $-F_{-2k-1} \le n < F_{-2k} - 1$ then, for some ℓ, $k > \ell > 0$,

$$-F_{-2k+2\ell-1} \le n - F_{-2k} < -F_{-2k+2\ell+1} < 0.$$

If $n = F_{-2k} - 1$,

$$n - F_{-2k} = F_{-2}.$$

Proof: If $-F_{-2k-1} \le n < F_{-2k} - 1$, then

$$1 < F_{-2k} - n \le F_{-2k} + F_{-2k-1} = F_{-2k+1},$$

so

$$1 < F_{-2k} - n \le F_{2k-1}$$

and, as in the proof of Lemma 1,

$$0 < F_{2k-2\ell-1} < F_{-2k} - n \le F_{2k-2\ell+1} \text{ where } k > \ell \ge 1,$$

i.e.,

$$-F_{-2k+2\ell-1} \le n - F_{-2k} < -F_{-2k+2\ell+1} < 0.$$

Algorithm Z: This algorithm produces, for a given integer, the promised sum of Fibonacci numbers.

(1) If $n = F_{-i}$ for some i, then stop.

(2) If $n > 0$ and for $k > 0$, $F_{2k} < n < F_{2k+1}$, i.e., $-F_{-2k} < n < F_{-2k-1}$, write $n = F_{-2k-1} + (n - F_{-2k-1})$, and apply this algorithm to $n - F_{-2k-1}$, giving the next term in the sum.

(3) If $n > 0$ and for $k > 0$, $F_{2k-1} < n < F_{2k}$, i.e., $F_{-2k+1} < n < -F_{-2k}$, write $n = F_{-2k+1} + (n - F_{-2k+1})$, and apply this algorithm to $n - F_{-2k+1}$, giving the next term in the sum.

(4) If $n < 0$ and for $k > 0$, $F_{2k-1} < -n < F_{2k}$, i.e., $F_{-2k} < n < -F_{-2k+1}$, write $n = F_{-2k} + (n - F_{-2k})$, and apply this algorithm to $n - F_{-2k}$, giving the next term in the sum.

(5) If $n < 0$ and for $k > 0$, $-F_{2k} < -n < F_{2k+1}$, i.e., $-F_{-2k-1} < n < F_{-2k}$, write $n = F_{-2k} + (n - F_{-2k})$, and apply this algorithm to $n - F_{-2k}$, giving the next term in the sum.

The algorithm terminates when, eventually,

$$n - F_{-i_1} - F_{-i_2} \cdots - F_{-i_m} = F_{-i_{m+1}}.$$

Lemma 5: Algorithm Z produces a representation of any nonzero integer n as a sum of Fibonacci numbers F_i where $i \le 0$ and any two of the i's differ by at least 2.

Proof: If after the application of (2), $n - F_{-2k-1} \ne F_{-j}$ for any j, we have, by Lemma 1:

$$-F_{-2k+2\ell-1} < n - F_{-2k-1} < -F_{-2k+2\ell+1} < 0, \text{ where } \ell > 0.$$

By applying (4) or (5), the algorithm next considers $n - F_{-2k-1} - F_{-2k+2\ell}$.
If after (3), $n - F_{-2k+1} \ne F_{-j}$, by Lemma 2:

$$0 < -F_{-2k+2\ell+2} < n - F_{-2k+1} \le -F_{-2k+2\ell}, \text{ where } \ell > 0.$$

By (2) or (3), the algorithm next considers $n - F_{-2k+1} - F_{-2k+2\ell+1}$.
If after (4), $n - F_{-2k} \ne F_{-j}$, by Lemma 3:

$$0 \le -F_{-2k+2\ell+2} < n - F_{-2k} < -F_{-2k+2\ell}, \text{ where } \ell > 0.$$

By (2) or (3) the algorithm next considers $n - F_{-2k} - F_{-2k+2\ell+1}$.
If after (5), $n - F_{-2k} \ne F_{-j}$, by Lemma 4:

$$-F_{-2k+2\ell-1} < n - F_{-2k} < -F_{-2k+2\ell+1} < 0, \text{ where } \ell > 0.$$

By (4) and (5), the algorithm next considers $n - F_{-2k} - F_{-2k+2\ell}$.
Thus, if the first stage of the algorithm produces $n - F_{-i}$ $(i > 0)$, the second produces $n - F_{-i} - F_{-i+p}$, where $p \ge 2$ and $-i + p < 0$.

The same applies to later stages of the algorithm which therefore produces Fibonacci numbers with subscripts at least two apart.

The next two lemmas are required to prove the uniqueness of this representation.

Lemma 6: (i) $\sum_{i=1}^{k} F_{-2i} = 1 - F_{-2k-1}$;

(ii) $\sum_{i=1}^{k} F_{-2i+1} = -F_{-2k}$;

(iii) $\sum_{i=1}^{k} F_{-i} = 1 - F_{-k+1}$.

Proof: The proof is simple and is therefore omitted here.

Lemma 7: If $i_1 > i_2 > \cdots > i_h > 0$ and, for $2 < j \le h$, $i_j - i_{j+1} \ge 2$,

$$-F_{-i_1+1} < \sum_{k=1}^{h} F_{-i_k} \le -F_{-i_1-1} \text{ if } i_1 \text{ is odd,}$$

and

$$-F_{-i_1-1} < \sum_{k=1}^{h} F_{-i_k} \le -F_{-i_1+1} \text{ if } i_1 \text{ is even.}$$

Proof: If i_1 is odd, by Lemma 6:

$$F_{-i_1} + F_{-i_1+3} + F_{-i_1+5} + \cdots + F_{-2} \le \sum_{k=1}^{h} F_{-i_k} \le F_{-i_1} + F_{-i_1+2} + \cdots + F_{-1}$$

$$F_{-i_1} + 1 - F_{-i_1+2} \le \sum_{k=1}^{h} F_{-i_k} \le -F_{-i_1-1},$$

so, $$-F_{-i_1+1} < 1 - F_{-i_1+1} \le \sum_{k=1}^{h} F_{-i_k} < -F_{-i_1-1}.$$

If i_1 is even, by Lemma 6:

$$F_{-i_1} + F_{-i_1+2} + \cdots + F_{-2} \le \sum_{k=1}^{h} F_{-i_k} \le F_{-i_1} + F_{-i_1+3} + \cdots + F_{-3} + F_{-1}$$

$$1 - F_{-i_1-1} \le \sum_{k=1}^{h} F_{-i_k} \le -F_{-i_1+1}.$$

Theorem 1: Algorithm Z expresses every integer n as a unique sum of a minimal number of distinct Fibonacci numbers F_i, where $i \le 0$.

Proof: If $n = 0$, $n = F_0$.

If $n \ne 0$, by Lemma 5 the algorithm produces a sum of the form

$$n = \sum_{k=1}^{h} F_{-i_k}, \text{ where } i_k \ge i_{k+1} + 2.$$

If the representation were not unique or not minimal, we would also have

$$n = \sum_{k=1}^{m} F_{-j_k}, \text{ where } j_k \ge j_{k+1} + 2, \text{ and possibly } m < h.$$

Let $-i_p$ and $-j_p$ be the first of these subscripts, if any, that are distinct and assume $i_p > j_p$. Then

$$n - F_{-i_1} - \cdots - F_{-i_{(p-1)}} = \sum_{k=p}^{h} F_{-i_k} = \sum_{k=p}^{m} F_{-j_k}.$$

If i_p and j_p are odd, then, by Lemma 7,

$$\sum_{k=p}^{h} F_{-} > -F_{-i_p+1} \quad \text{and} \quad -F_{-j_p-1} \ge \sum_{k=p}^{m} F_{-j_k}.$$

Also, $i_p - 2 \ge j_p$, and so $-F_{-i_p+1} \ge -F_{-j_p-1}$, which is impossible.

If i_p is odd and j_p is even, then

$$\sum_{k=p}^{h} F_{-i_k} \text{ is positive and } \sum_{k=p}^{m} F_{-j_k} \text{ is negative}$$

by Lemma 7.

Similarly, if i_p is even and j_p is odd, then

$$\sum_{k=p}^{h} F_{-i_k} \text{ is negative and } \sum_{k=p}^{m} F_{-j_k} \text{ is positive}$$

by Lemma 7.

If i_p and j_p are both even, then $i_p - 2 \ge j_p$, and by Lemma 7,

$$\sum_{k=p}^{h} F_{-i_k} \le -F_{i_p+1} \quad \text{and} \quad -F_{-i_p-1} < \sum_{k=p}^{m} F_{-j_k}$$

and also

$$-F_{i_p+1} \le -F_{-j_p-1},$$

which is impossible.

Thus, for $1 \le k \le m$, $i_k = j_k$.

If $m < h$, we have by the above:

$$n = \sum_{k=1}^{m} F_{-i_k} = \sum_{k=1}^{h} F_{-i_k},$$

so

$$\sum_{k=m+1}^{h} F_{-i_k} = 0.$$

If $h > m + 1$, then by Lemma 7, if i_{m+1} is odd, $-F_{-i_{m+1}+1} < 0$, and if i_{m+1} is even, then $0 \le -F_{-i_{m+1}+1}$, both of which are impossible.

If $h = m + 1$, then $F_{-i_h} = 0$, which is impossible because $i_h \neq 0$.

Therefore, the representation of n is unique and minimal.

As any representation of a number n as a sum of Fibonacci numbers

$$\sum_{k=1}^{h} F_{-i_k}, \text{ where } i_1 > i_2 > \cdots > i_h > 0,$$

can be changed to

$$\sum_{k=1}^{h-1} F_{-i_k} + F_{-i_h-1} + F_{-i_h-2},$$

it is clear that there can be no maximal number of Fibonacci numbers in a given sum.

References

1. J. L. Brown. "Zeckendorf's Theorem and Some Applications." *Fibonacci Quarterly* *2.2* (1964):163-68.
2. J. L. Brown. "A New Characterization of the Fibonacci Numbers." *Fibonacci Quarterly* *3.1* (1975):1-8.

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Author and Title Index for *The Fibonacci Quarterly*

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for *The Fibonacci Quarterly.* In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of *The Fibonacci Quarterly.* Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in *The Fibonacci Quarterly.* We would deeply appreciate it if all authors of articles published in *The Fibonacci Quarterly* would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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