ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to **Dr. A. P. Hillman** in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-712 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that for all positive integers n, $\alpha(\sqrt{5}\alpha^n - L_{n+1})$ is a Lucas number.

B-713 *Proposed by Herta T. Freitag, Roanoke, VA*

Dedicated to Dr. A. P. Hillman

(a) Let S be a set of three consecutive Fibonacci numbers. In a Pythagorean triple, (a, b, c), a is the product of the elements in S; b is the product of two Fibonacci numbers (both larger than 1), one of them occurring in S; and c is the sum of the squares of two members of S. Determine the Pythagorean triple and prove that the area of the corresponding Pythagorean triangle is the product of four consecutive Fibonacci numbers.

(b) Same problem as part (a) except that Fibonacci numbers are replaced by Lucas numbers.

B-714 *Proposed by J. R. Goggins, Whiteinch, Glasgow, Scotland*

Dedicated to Dr. A. P. Hillman

Define a sequence G_n by $G_0 = 0$, $G_1 = 4$, and $G_{n+2} = 3G_{n+1} - G_n - 2$ for $n \geq 0$. Express G_n in terms of Fibonacci and/or Lucas numbers.

B-715 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Dedicated to Dr. A. P. Hillman

Prove that if $s > 2$,

$$F_m \equiv 0 \pmod{F_s^2} \text{ if and only if } m \equiv 0 \pmod{sF_s}.$$

B-716 *Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA*

Dedicated to Dr. A. P. Hillman

If a and b have the same parity, prove that $L_a + L_b$ cannot be a prime larger than 5.

B-717 *Proposed by L. Kuipers, Sierre, Switzerland*

Show that

$$\arctan\frac{2}{5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{L_{2n+1}}{2^{2n+1}}.$$

SOLUTIONS

Edited by A. P. Hillman

Differences in {1, 2}

B-688 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Find the number of increasing sequences of integers such that 1 is the first term, n is the last term, and the difference between successive terms is 1 or 2. [For example, if $n = 8$, then one such sequence is 1, 2, 3, 5, 6, 8 and another is 1, 3, 4, 6, 8.]

Solution by H.-J. Seiffert, Berlin, Germany

Let A_n denote the set of all sequences having the desired properties. If $|A_n|$ denotes the number of elements of A_n, then clearly $|A_1| = |A_2| = 1$. We shall prove that $|A_n| = F_n$. This is true for $n = 1, 2$. Assume that it holds for all $k \in \{1, \ldots, n-1\}$ $(n \geq 3)$. If B_n and C_n denote the set of all sequences of A_n, where the second term from the right equals $n - 2$ and $n - 1$, respectively, then we obviously have

$$|B_n| = |A_{n-2}| \quad \text{and} \quad |C_n| = |A_{n-1}|.$$

Since $A_n = B_n \cup C_n$ and $B_n \cap C_n = \emptyset$, the induction hypothesis gives

$$|A_n| = |B_n| + |C_n| = |A_{n-2}| + |A_{n-1}| = F_{n-2} + F_{n-1} = F_n.$$

This completes the induction proof.

Also solved by Charles Aschbacher, Glenn Bookhout, Paul S. Bruckman, Russell Jay Hendel, Douglas E. Iannucci, Norbert Jensen, Carl Libis, Ray Melham, Bob Prielipp, Sahib Singh, Lawrence Somer, Shanon Stamp, and the proposer.

Numbers with Even Zeckendorf Representations

B-689 *Proposed by Philip L. Mana, Albuquerque, NM*

Show that $F_n^2 - 1$ is a sum of Fibonacci numbers with distinct positive even subscripts for all integers $n \geq 3$.

Solution by Lawrence Somer, Washington, D.C.

It follows by identity I_{10} on page 56 of Hoggatt's *Fibonacci and Lucas Numbers* that

$$F_n^2 - F_{n-2}^2 = F_{2n-2}.$$

Thus,

$$F_n^2 - 1 = F_{2n-2} + (F_{n-2}^2 - 1).$$

The result now follows by induction upon noting that

$$F_3^2 - 1 = 2^2 - 1 = 3 = F_4 \quad \text{and} \quad F_4^2 - 1 = 3^2 - 1 = 8 = F_6.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Russell Jay Hendel, Douglas E. Iannucci, Norbert Jensen, Joseph J. Kostal, Ray Melham, Alex Necochea, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Golden Geometric Progressions

B-690 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k = \alpha^{10k+1} + \alpha^{10k+2} + \alpha^{10k+3} + \cdots + \alpha^{10k+10}$, where $\alpha = (1 + \sqrt{5})/2$. Find positive integers b and c such that $S_k/\alpha^{10k+b} = c$ for all nonnegative integers k.

Solution by Paul S. Bruckman, Edmonds, WA

$$S_k = \sum_{i=1}^{10} \alpha^{10k+i} = \alpha^{10k}\left(\frac{\alpha^{11} - \alpha}{\alpha - 1}\right) = \frac{\alpha^{10k+1}}{\alpha^{-1}}(\alpha^{10} - 1)$$

$$= \alpha^{10k+2} \cdot \alpha^5(\alpha^5 + \beta^5) = 11\alpha^{10k+7}.$$

We see that the values $b = 7$, $c = 11$ solve the problem.

Also solved by Tareq Al-Naffouri, Glenn Bookhout, Russell Jay Hendel, Norbert Jensen, Ray Melham, Bob Prielipp, H.-J. Seiffert, Sahib Singh, and the proposer.

Rectangles in Similar Rectangles

B-691 *Proposed by Heiko Harborth, Technische U. Braunschweig, W. Germany*

Herta T. Freitag asked whether a golden rectangle can be inscribed into a larger golden rectangle (all four vertices of the smaller are points on the sides of the larger one). An answer follows from the solution of the generalized problem: Which rectangles can be inscribed into larger similar rectangles?

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

All nonsquare rectangles can be inscribed in larger similar rectangles.

Indeed, suppose a given rectangle, R_1, has sides a and b with $a \neq b$. Set

$$c = \{\max(a, b)\}^2/\{\min(a, b)\}$$

and consider the rectangle, R_2, with sides $\max(a, b)$ and c.

R_2 is similar to R_1, because

$$c/\max(a, b) = \max(a, b)/\min(a, b),$$

R_1 can be inscribed in R_2, with common side $\max(a, b)$, because $c \geq \min(a, b)$, and R_2 is larger than R_1 because $c > \neq\min(a, b)$ when $a \neq b$. This completes the proof.

Editor's Note: The proposer made the tacit assumption that all the vertices of the smaller rectangle are *interior* points of the sides of the larger one. With this assumption, Paul S. Bruckman, Herta T. Freitag, and the proposer showed that the rectangles have to be squares.

A Fibonacci Factorization

B-692 *Proposed by Gregory Wulczyn, Lewisburg, PA*

Let $G(a, b, c) = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}$. Prove or disprove that each of F_{a+b+c}, F_{b+c-a}, F_{c+a-b}, and F_{a+b-c} is an integral divisor of $G(a, b, c)$ for all odd positive integers a, b, and c.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

We prove the stronger assertion

$$25F_{a+b+c}F_{a+b-c}F_{a-b+c}F_{c+b-a} = L_{2a}L_{2b}L_{2c} + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4.$$

The proof is verbatim identical to the published solution to B-669, Vol. 29, no. 2, p. 185, with, however, the word *odd* replaced by *even*.

"From the identity

$$5F_{m+n}F_{m-n} = L_{2m} - (-1)^{m+n}L_{2n}$$

we get [setting $(-1)^{a+b+c} = \mathrm{e}$)

$$\begin{aligned} 25F_{a+b+c}F_{a+b-c}F_{a-b+c}F_{c+b-a} &= [L_{2a+2b} - \mathrm{e}L_{2c}][L_{2c} - \mathrm{e}L_{2a-2b}] \\ &= L_{2c}[L_{2a+2b} + L_{2a-2b}] - \mathrm{e}L_{2c}^2 - \mathrm{e}L_{2a+2b}L_{2a-2b} \\ &= L_{2c}[L_{2a}L_{2b}] - \mathrm{e}L_{2c}^2 - \mathrm{e}[L_{4a} + L_{4b}] \\ &= L_{2a}L_{2b}L_{2c} - \mathrm{e}[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4], \end{aligned}$$

and for a, b, and c odd (actually for $a + b + c$ odd), the given identity is established."

Also solved by Paul S. Bruckman, Norbert Jensen, Bob Prielipp, and the proposer.

A Combinatorial Problem

B-693 *Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Tech, Atlanta, GA*

Let A consist of all pairs $\{x, y\}$ chosen from $\{1, 2, \ldots, 2n\}$, B consist of all pairs from $\{1, 2, \ldots, n\}$, and C of all pairs from $\{n+1, n+2, \ldots, 2n\}$. Let S consist of all sets $T = \{P_1, P_2, \ldots, P_k\}$ with the P_i (distinct) pairs in A. How many of the T in S satisfy at least one the the conditions:

(i) $P_i \cap P_j \neq \emptyset$ for some i and j, with $i \neq j$,

(ii) $P_i \in B$ for some i, or

(iii) $P_i \in C$ for some i?

Solution by Philip L. Mana, Albuquerque, NM

Let $C(n, k)$ denote $\binom{n}{k}$. There are $C(2n, 2)$ pairs in A; thus $C(C(2n, 2), k)$ sets T in S. The sets T meeting none of the conditions (i), (ii), (iii) can be written in the form

$$U = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}\}$$

with the x_i a set of k integers chosen from $\{1, 2, \ldots, n\}$ and satisfying

$$x_1 < x_2 < \cdots < x_k$$

and the y_i a permutation of k distinct integers from $\{n+1, n+2, \ldots, 2n\}$. The number of such sets U is

$$C(n, k)P(n, k) = k!\binom{n}{k}^2;$$

hence, the number of sets T satisfying at least one of the conditions is

$$C(C(2n, 2), k) - C(n, k)P(n, k).$$

Also solved by the proposers, who indicated that the problem arose in a combinatorial study in parallel processing.

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