k-REVERSE MULTIPLES

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We begin with the simple observation that $4 \cdot (2178) = 8712$. That is, when 2187 is multiplied by 4, the result is 8712 which is 2178 with the digits reversed. Since 4 is the multiplier that produces the reversal of digits, we call 2178 a *4-reverse multiple*. More generally, let x be an n-digit, base g number

$$(1) \qquad x = \sum_{i=0}^{n-1} a_i g^i$$

with $0 \le a_i < g$ and $a_{n-1} \ne 0$. Then x is called a *k-reverse multiple* if, for some integer k, $1 < k < g$,

$$(2) \qquad kx = \sum_{i=0}^{n-1} a_{n-1-i} g^i.$$

Previously, most work on k-reverse multiples has focused on either finding all those less than a given m [1], or characterizing, for a given n, those with n-digits. This latter problem seems to be quite difficult and has been completely solved only for the 2- and 3-digit cases (see [1] and [3]). Additionally, various schemes have been advanced for calculating these multiples (see [2] and [3]). Beyond this, it has been noted that once a k-reverse multiple is known, it may be used to create others. For example, it is easily verified that 21782178 and 21978 are also base 10, 4-reverse multiples.

What has not been discussed previously is how to find *all* k-reverse multiples once those with a small number of digits are known. For example, in base 11, 118 and 1298 are 7-reverse multiples. While it is clear that 118118 is a 7-reverse multiple, it is not as obvious that 11918 is also such a multiple. This question, of how to form multiples having a large number of digits from those with a small number, is the focus of our discussion. As we will see, the solution has a graphic representation.

We begin by supposing that x is an n-digit, base g, k-reverse multiple. From (1) and (2), we obtain the following set of equations by comparing corresponding digits of kx:

$$(3) \qquad \begin{array}{lllll} ka_0 & & = a_{n-1} & + r_0 g \\ ka_1 & + r_0 & = a_{n-2} & + r_1 g \\ & & \;\;\cdots & \\ ka_i & + r_{i-1} & = a_{n-1-i} & + r_i g \\ & & \;\;\cdots & \\ ka_{n-2} & + r_{n-3} & = a_1 & + r_{n-2} g \\ ka_{n-1} & + r_{n-2} & = a_0 & \end{array}$$

where $0 \le r_i < g$ for $i = 0, \ldots, n-2$. The last equation implies $a_0 \ne 0$ since $a_{n-1} \ne 0$. The r_i's are the so-called "carry numbers." As we will see, these numbers determine the character of k-reverse multiples.

To determine whether there are any k-reverse multiples for a given g and k, we consider the equations in (3) two at a time. For convenience, let $r_{-1} = r_{n-1} = 0$. At the $(i+1)^{\text{st}}$ step, $i = 0, 1, \ldots$, we examine the pair of equations

$$(4) \qquad \begin{cases} ka_i + r_{i-1} = a_{n-1-i} + r_i g \\ ka_{n-1-i} + r_{n-2-i} = a_i + r_{n-1-i} g \end{cases}$$

where r_{i-1} and r_{n-1-i} are known from the previous step. That is, we seek nonnegative integers a_i, a_{n-1-i}, r_i, and r_{n-2-i} which, in addition to (4), satisfy

$$(5) \qquad \begin{cases} 0 < a_0,\ a_{n-1} \\ a_i < g,\ i = 0, 1, \ldots, n-1 \end{cases}$$

and

$$(6) \qquad r_i < g,\ i = 0, 1, \ldots, n-2.$$

The equations in (4) along with the inequalities in (5) imply tighter restrictions in (6). That is the content of the following lemma.

Lemma 1: Suppose there exist nonnegative integers which satisfy (4) and (5) for $i = 0, 1, \ldots, n-1$. Then the following hold:

$$(7) \qquad \begin{cases} 0 < r_0 \\ r_i < k \text{ for } i = 0, \ldots, n-2. \end{cases}$$

Proof: Solving (4) for a_i gives

$$(8) \qquad a_i = (kr_i g - kr_{i-1} + r_{n-1-i} g - r_{n-2-i})/(k^2 - 1).$$

Letting $i = 0$ and using $r_{-1} = r_{n-1} = 0$ in (8) gives

$$a_0(k^2 - 1) = kr_0 g - r_{n-2}.$$

Hence, $0 < r_0$ since $1 < k$, $0 < a_0$, and $0 \le r_{n-2}$.

To show the second part of (7), suppose $r_{i-1} < k$; note that when $i = 0$, the supposition is valid. Then from the general equation in (3) we have

$$r_i g \le ka_i + r_{i-1} < ka_i + k = k(a_i + 1) \le kg$$

and hence $r_i < k$. □

One convenient way to proceed is to look for nonnegative integers a_i and a_{n-1-i} satisfying (5) such that

$$(9) \qquad \begin{cases} ka_i + r_{i-1} \equiv a_{n-1-i} \pmod{g} \\ 0 \le a_i + r_{n-1-i} g - ka_{n-1-i} < k \end{cases}$$

where r_{i-1} and r_{n-1-i} have been determined in the previous step. If the a's exist, then r_i and r_{n-2-i} can be found by (4):

$$(10) \qquad \begin{cases} r_i = (ka_i + r_{i-1} - a_{n-1-i})/g \\ r_{n-2-i} = a_i + r_{n-1-i} g - ka_{n-1-i}. \end{cases}$$

The restrictions in (9) guarantee the r's in (10) are nonnegative.

The above procedure is successful when, at each step, there are nonnegative integers a_i and a_{n-1-i} which satisfy (5) and (9). The following graphical notation will be convenient. If r_{n-1-i}, r_{i-1}, a_{n-1-i}, a_i, r_{n-2-i}, and r_i satisfy (4), (5), and (7), then we will write

$$(11) \qquad \begin{array}{l} (r_{n-1-i},\ r_{i-1}) \\ \quad\Big|\quad (a_{n-1-i},\ a_i) \\ (r_{n-2-i},\ r_i) \end{array}$$

and conversely. Thus, we hope to generate a graph, or more precisely, a rooted tree in which a path from the root to a node has the following form and labels:

$$
(12)\qquad
\begin{array}{l}
(0,\ 0)\\
\quad\big|\ (a_{n-1},\ a_0)\\
(r_{n-2},\ r_0)\\
\quad\big|\ (a_{n-2},\ a_1)\\
(r_{n-3},\ r_1)\\
\quad|\\
\quad\cdots\\
\quad|\\
(r_{n-1-i},\ r_{i-1})\\
\quad\big|\ (a_{n-1-i},\ a_i)\\
(r_{n-2-i},\ r_i)
\end{array}
$$

We will use this notation in the examples below. Since $0 \le r_i < k$, there can be at most k^2 different pairs of r_i's used as node labels in the tree. If a node is labeled with an r-pair that has already appeared in the tree, the tree can be pruned after this node, since no new information will be obtained beyond this point. When needed for analysis, a pruned tree can be extended by replicating earlier sections of it. Before proceeding further with the exposition, we look at the tree for the 4-reverse multiple, 2178.

Example 1: $g = 10$, $k = 4$.

Let us begin by considering (9) with $i = 0$. The various possibilities are:

(13)

a_0	$a_{n-1} \equiv 4a_0$	$a_0 - 4a_{n-1}$
1	4	-15
2	8	-30
3	2	-5
4	6	-20
5	0	5
6	4	-10
7	8	-25
8	2	0
9	6	-15

Only $a_0 = 8$ satisfies the required condition

$$0 \le a_0 - 4a_{n-1} < k = 4.$$

Using (10), it can be shown that $r_{n-2} = 0$ and $r_0 = 3$. Continuing in this manner and using the above notation, the following is obtained:

(14)

```
              (0, 0)
                 | (2, 8)
              (0, 3)
                 | (1, 7)
              (3, 3)
      (7, 1)  /     \  (9, 9)
         (3, 0)    (3, 3)
      (8, 2) |
         (0, 0)
 (0, 0)  /     \  (2, 8)
    (0, 0)    (0, 3)
```

The tree is not continued any further since (0, 0), (0, 3), and (3, 3) have appeared previously. The careful reader will observe that

$$\begin{array}{c}(0,\ 0)\\ \Big|\,{\scriptstyle (0,\ 0)}\\ (0,\ 0)\end{array}$$

appears at the end of the tree, but not initially. This will always be the case since the equations in (4) are satisfied by the trivial or zero solution. Although $r_0 \neq 0$, the r-pair (0, 0) is permissible after the first step. □

The following question arises immediately. How do we use such a tree to find k-reverse multiples? The next two theorems provide answers.

Theorem 1: For a given g and k, suppose a tree of the form (12) exists; that is, suppose nonnegative solutions to (4), (5), and (7) exist. Then there is an $n = 2i + 2$-digit number satisfying (2) if and only if $r_{n-2-i} = r_i$. In this case, x is given by

(15) $$x = a_{n-1}a_{n-2} \ldots a_{n-1-i}a_i \ldots a_1a_0.$$

Proof: In forming (12), the equations to be considered at the $(i+1)^{\text{st}}$ step are

(16) $$\begin{cases} ka_i + r_{i-1} = a_{n-1-i} + \mathbf{r_i}\, g \\ ka_{n-1-i} + r_{n-2-i} = a_i + \mathbf{r_{n-1-i}}\, g \end{cases}$$

The two quantities in bold type are the r's to be determined at this step. If $n = 2i + 2$, then this is the last set of equations to be considered. Since

$$n - 2 - i = (2i + 2) - 2 - i = i,$$

$r_{n-2-i} = r_i$ and the conclusion follows. Conversely, if $r_{n-2-i} = r_i$, then we may stop with (16) by letting $n - 2 - i = i$ to give $n = 2i + 2$. □

Corollary 1: For base g, suppose there are k-reverse multiples. Let n be an even number. Then there exists an n-digit multiple if and only if the corresponding infinite tree contains a path of length $n/2$ from the root to a node designated by (u, u).

Proof: This is simply a restatement of Theorem 1. □

Example 1 continued: By Corollary 1, to find all base 10, 4-reverse multiples with an even number of digits, we traverse the tree in (14) stopping at nodes of the form (u, u). Thus, we see that (3, 3) gives rise to a 4-digit multiple. To find this number, we use (15) of Theorem 1. We read it off from the a-pairs, starting at the root, reading down the left-hand side and then back up the right. Thus, we find that 2178 is a 4-reverse multiple. So, too, are the following numbers:

219978, 21782178, 21999978,
2178002178, 2197821978, 2199999978.

Of course, there are infinitely many, but these are the ones with the least number of even digits. It should be remembered that the tree is actually infinite, and that pruned branches may be extended when needed to obtain additional desired numbers.

Theorem 2: For a given g and k, suppose a tree of the form (12) exists; that is, suppose nonnegative solutions to (4), (5), and (7) exist. Then there is an $n = 2i + 3$-digit number satisfying (2) if and only if

$$(k - 1) \mid (r_{n-2-i}g - r_i) \quad \text{and} \quad 0 \le (r_{n-2-i}g - r_i)/(k - 1) < g.$$

In this case, x is given by

(17) $\quad x = a_{n-1}a_{n-2} \ldots a_{n-1-i}Ma_i \ldots a_1a_0$

where $M = (r_{n-2-i}g - r_i)/(k - 1)$.

Proof: If $n = 2i + 3$, then there are an odd number of equations in (3). After $(i+1)$ steps, we are left with

(18) $\quad ka_{i+1} + r_i = a_{n-2-i} + r_{i+1}g.$

Since $n - 2 - i = i + 1$, (18) becomes

(19) $\quad ka_{i+1} + r_i = a_{i+1} + r_{n-2-i}g.$

Because r_i and r_{n-2-i} are already known, we must have

(20) $\quad a_{i+1} = (r_{n-2-i}g - r_i)/(k - 1).$

Thus, after determining r_{n-2-i} and r_i, we can stop if and only if

$$(k - 1)\,|\,(r_{n-2-i}g - r_i) \quad \text{and} \quad 0 \le (r_{n-2-i}g - r_i)/(k - 1) < g.$$

When this occurs, x is given by (17). □

In order to apply Theorem 2 to a tree, we must check at each step to see if $(k - 1)\,|\,(r_{n-2-i}g - r_i)$ and $0 \le (r_{n-2-i}g - r_i)/(k - 1) < g$. Thus, in Example 1, since $3\,|\,(3 \cdot 10 - 3)$ and $0 \le (3 \cdot 10 - 3)/3 < 10$, the r-pair (3, 3) yields the 4-reverse multiple 21978. The following theorem simplifies this tedious checking process.

Theorem 3: For a given g and k, suppose a tree of the form (12) exists; that is, suppose nonnegative solutions to (4), (5), and (7) exist. Then there is an $n = 2i + 3$-digit number satisfying (2) if and only if the graph contains

$$\begin{array}{l} (r_{n-1-i},\ r_{i-1}) \\ \Big| \quad (a_{n-1-i},\ a_i) \\ (r_{n-2-i},\ r_i) \\ \Big| \quad (a_{n-2-i},\ a_{i+1}) \\ (r_i,\ r_{n-2-i}). \end{array}$$

Further, when this occurs, $a_{n-2-i} = a_{i+1} = M = (r_{n-2-i}g - r_i)/(k - 1)$.

$$a_{n-2-i} = a_{i+1} = M = (r_{n-2-i}g - r_i)/(k - 1).$$

The desired n-digit number x is given by (17).

Proof: Suppose there is a $2i + 3$-digit k-reverse multiple. The first piece of the above graph exists by assumption. We must show the existence of the second piece. Equations (4) at the $(i+2)^{\text{nd}}$ step are

$$(21) \quad \begin{cases} ka_{i+1} + r_i = a_{n-2-i} + r_{i+1}g \\ ka_{n-2-i} + r_{n-3-i} = a_{i+1} + r_{n-2-i}g. \end{cases}$$

From (19) and (20) in the proof of Theorem 2, we have

$$kM + r_i = M + r_{n-2-i}g.$$

Thus, one solution to (21) is

$$\begin{array}{ll} a_{n-2-i} = M; & a_{i+1} = M \\ r_{n-3-i} = r_i; & r_{i+1} = r_{n-2-i} \end{array}$$

and the result follows.

Now suppose for a given g and k there exists a graph containing

$$
\begin{tikzcd}[arrows=-]
(r_{n-2-i},\ r_i) \arrow[d, "{(a_{n-2-i},\ a_{i+1})}"] \\
(r_i,\ r_{n-2-i}).
\end{tikzcd}
$$

By hypothesis, (4) becomes

$$
\begin{cases} ka_{i+1} + r_i = a_{n-2-i} + r_{n-2-i}g \\ ka_{n-2-i} + r_i = a_{i+1} + r_{n-2-i}g. \end{cases} \tag{22}
$$

Subtracting one equation from the other gives $a_{i+1} = a_{n-2-i}$. From this, it follows that $(k - 1) | (r_{n-2-i}g - r_i)$ and $a_{n-2-i} = a_{i+1} = (r_{n-2-i}g - r_i)/(k - 1)$, so by Theorem 2 there exists an $n = 2i + 3$-digit k-reverse multiple. □

Corollary 2: For base g, suppose there are k-reverse multiples. Let n be an odd number. Then there exists an n-digit multiple if and only if the corresponding infinite tree contains a path of length $(n - 1)/2$ from the root to nodes designated by (u, v) followed by (v, u).

Proof: This is simply a restatement of Theorem 3. □

The importance of Corollaries 1 and 2 cannot be overstated. Suppose it is known that for a given g there are k-reverse multiples. Then we use the procedure suggested by (9) to create a pruned tree. By traversing the tree, replicating earlier sections when necessary, and stopping at those pairs which have the form given in the above corollaries, we are able to find all k-reverse multiples for a given n. This procedure is illustrated in the following example.

Example 2: $g = 19$, $k = 14$.

The tree in this case is:

$$
\begin{tikzcd}[arrows=-]
 & & & (0,\ 0) \arrow[d, "{(1,\ 15)}"] & \\
 & & & (1,\ 11) \arrow[d, "{(2,\ 17)}"] & \\
 & & & (8,\ 13) \arrow[d, "{(11,\ 8)}"] & \\
 & & & (6,\ 6) \arrow[d, "{(8,\ 11)}"] & \\
 & & & (13,\ 8) \arrow[dl, "{(17,\ 2)}"'] \arrow[dr, "{(18,\ 17)}"] & \\
 & & (11,\ 1) \arrow[dl, "{(15,\ 1)}"'] \arrow[dr, "{(16,\ 16)}"] & & (12,\ 12) \arrow[d, "{(17,\ 18)}"] \\
 & (0,\ 0) \arrow[dl, "{(0,\ 0)}"'] \arrow[dr, "{(1,\ 15)}"] & & (1,\ 11) & (8,\ 13) \\
(0,\ 0) & & (1,\ 11) & &
\end{tikzcd}
$$

By Corollaries 1 and 2, we can traverse the tree stopping at (6, 6), (12, 12), (11, 1), and (0, 0). The first two nodes give 14-reverse multiples with an even number of digits, while the third gives rise to those with an odd number of digits. The pair (0, 0) of course always accounts for multiples with both an even and an odd number of digits. So there are 6-, 10-, 11-, 12-, ...-digit 14-reverse multiples.

Those with the least number of digits are:

1 2 11 8 17 15
1 2 11 8 18 17 11 8 17 15
1 2 11 8 17 16 2 11 8 17 15

It would be difficult, using these, to see that

1 2 11 8 18 17 11 8 18 17 11 8 17 15

and

1 2 11 8 17 16 2 11 8 18 17 11 8 17 16 2 11 8 17 15

are also k-reverse multiples. Yet, using the tree, it is clear that they are.□

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