ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems

PROBLEM PROPOSED IN THIS ISSUE

H-474 ***Proposed by R. André-Jeannin, Longwy, France***

Let us define the sequence $\{U_n\}$ by

$$U_0 = 0,\ U_1 = 1,\ U_n = PU_{n-1} - QU_{n-2},\ n \in Z,$$

where P and Q are nonzero integers. Assuming that $U_k \neq 0$, the matrix M_k is defined by

$$M_k = \frac{1}{U_k}\begin{pmatrix} U_{k+1} & iQ^{k/2} \\ iQ^{k/2} & -Q^k U_{1-k} \end{pmatrix},\quad k \geq 1,$$

where $i = \sqrt{(-1)}$.

Express in a closed form the matrix M_k^n, for $n \geq 0$.

Reference: A. F. Horadam & P. Filipponi. "Choleski Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *Fibonacci Quarterly* **29.2** (1991):164-73.

SOLUTIONS

How Many?

H-456 ***Proposed by David Singmaster, Polytechnic of the South Bank, London, England*** *(Vol. 29, no. 3, August 1991)*

Among the Fibonacci numbers, F_n, it is known that: 0, 1, 144 are the only squares; 0, 1, 8 are the only cubes; 0, 1, 3, 21, 55 are the only triangular numbers. [See Luo Ming's article in *The Fibonacci Quarterly* **27.2** (1989):98-108.]

A. Let $p(m)$ be a polynomial of degree at least 2 in m. Is it true that $p(m) = F_n$ has only finitely many solutions?

B. If we replace F_n by an arbitrary recurrent sequence f_n, we cannot expect a similar result, since f_n, can easily be a polynomial in n. Even if we assume the auxiliary equation of our recurrence has no repeated roots, we still cannot expect such a result. For example, if

$$f_n = 6f_{n-1} - 8f_{n-2},\ f_0 = 2,\ f_1 = 6,$$

then

$$f_n = 2^n + 4^n,$$

so every f_n is of the form $p(m) = m^2 + m$. What restriction(s) on f_n, is(are) needed to make $f_n = p(m)$ have only finitely many solutions?

Comments: The results quoted have been difficult to establish, so part A is likely to be quite hard and, hence, part B may well be extremely hard.

Solution by Paul S. Bruckman, Edmonds WA

To simplify the problem somewhat, we assume that $(f_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers, and that the f_n's satisfy a homogeneous linear recurrence of order d $(d \geq 2)$. Furthermore, we assume that the roots of the characteristic equation of f_n are distinct. Let these roots be denoted by z_j, $j = 1, 2, \ldots, d$, with $|z_1| \leq |z_2| \leq \cdots \leq |z_d|$. Then constants a_j exist such that

$$(1) \qquad f_n = \sum_{j=1}^{d} a_j z_j^n, \; n = 0, 1, 2, \ldots .$$

We shall also suppose that the sequence $(p(n))_{n=1}^{\infty}$ is an increasing sequence of positive integers from some point on. Let e denote the degree of p $(e \geq 2)$. Then constants b_j exist such that

$$(2) \qquad p(z) = \sum_{j=0}^{e} b_j z^j .$$

Under these assumptions, we shall prove the following

Theorem: $f_n = p(m_n)$ for infinitely many n, where the m_n's are positive integers, if and only if $f_n = p(z_1^n)$ for all n. If these conditions are met, we must also have:

(i) $p(0) = 0$;
(ii) $d = e$;
(iii) z_1 is an integer > 1;
(iv) $z_j = z_1^j$, $j = 1, 2, \ldots, d$.

Proof: If $f_n = p(z_1^n)$ for all n, clearly $f_n = p(m_n)$ for infinitely many n, with $m_n = z_1^n$. Conditions (i), (ii), (iii), and (iv) must then follow.

Conversely, suppose $f_n = p(m_n)$ for infinitely many n, for some sequence $(m_n)_{n=1}^{\infty}$ of positive integers. Then, for some subsequence $(n_k)_{k=1}^{\infty}$ of positive integers, we must have

$$(3) \qquad f_{n_k} = p(m_{n_k}), \; k = 1, 2, \ldots .$$

Given any $e+2$ consecutive elements $n_{1+t}, n_{2+t}, \ldots, n_{e+2+t}$ $(t = 0, 1, 2, \ldots)$, we may form the $(e+1)^{\text{th}}$ divided difference of p with respect to $m_{n_{1+t}}, m_{n_{2+t}}, \ldots, m_{n_{e+2+t}}$. Since p is a polynomial, this expression must vanish. Thus $\Delta^{e+1} m_{n_{1+t}}, m_{n_{2+t}}, \ldots, m_{n_{e+2+t}}(p) = 0$, or

$$(4) \qquad \sum_{k=1}^{e+2} c_{k,t}\, p\left(m_{n_{k+t}}\right) = 0, \; t = 0, 1, 2, \ldots,$$

where

(5) $$c_{k,t} = \prod_{\substack{j=1 \\ j \neq k}}^{e+2} \left(m_{n_{k+t}} - m_{n_{j+t}}\right)^{-1}.$$

Then,

$$\sum_{k=1}^{e+2} c_{k,t} \sum_{j=0}^{e} b_j \left(m_{n_{k+t}}\right)^j = \sum_{j=0}^{e} b_j \sum_{k=1}^{e+2} c_{k,t} \left(m_{n_{k+t}}\right)^j = 0.$$

Since this is true for all $t \geq 0$, and the b_j's are assumed not to all equal 0, it follows that

(6) $$\sum_{k=1}^{e+2} c_{k,t} \left(m_{n_{k+t}}\right)^j = 0, \quad t = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, e.$$

On the other hand, due to (3), we also have the following:

$$\sum_{k=1}^{e+2} c_{k,t} f_{n_{k+t}} = 0, \text{ or } \sum_{k=1}^{e+2} c_{k,t} \sum_{j=1}^{d} a_j z_j^{n_{k+t}} = \sum_{j=1}^{d} a_j \sum_{k=1}^{e+2} c_{k,t} z_j^{n_{k+t}} = 0.$$

Again, since this is true for all $t \geq 0$, and all the a_j's are assumed not all equal to 0, we must have:

(7) $$\sum_{k=1}^{e+2} c_{k,t} z_j^{n_{k+t}} = 0, \quad t = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, d.$$

Comparing (6) and (7), since these are true for all $t \geq 0$, the two expressions must be **identically** equal. Therefore, the following is implied:

(8) $$b_0 = 0; \; d = e; \; \left(m_{n_{k+t}}\right)^j = z_j^{n_{k+t}}, \quad t = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, d.$$

We see that (8) implies conditions (i)-(iv) of the Theorem. As a result, we have:

(9) $$f_{n_k} = p\left(z_1^{n_k}\right), \quad k = 1, 2, \ldots .$$

Thus,

$$\sum_{j=1}^{d} a_j z_1^{jn_k} = \sum_{j=1}^{d} b_j \left(m_{n_k}\right)^j = \sum_{j=1}^{d} b_j z_1^{jn_k}.$$

Using the same argument as before (with k replacing t), it follows that

(10) $$a_j = b_j, \quad j = 1, 2, \ldots, d.$$

Therefore, for **all** n,

$$f_n = \sum_{j=1}^{d} a_j z_1^{nj} = \sum_{j=1}^{d} b_j z_1^{nj},$$

or

(11) $$f_n = p(z_1^n), \quad n = 0, 1, 2, \ldots .$$

Note that $m_n = z_1^n$ for all n; since the m_n's are to be integers, it must follows that z_1 is an integer. Also, since $(m_n)_{n=1}^{\infty}$ is increasing from some point on, we must have $z_1 > 1$; in fact, $(m_n)_{n=1}^{\infty}$ is increasing for **all** n. This completes the proof of the Theorem.

We can now readily dispose of the problem. Since $F_n = 5^{-\frac{1}{2}}(\alpha^n - \beta^n) = 5^{-\frac{1}{2}}[(-1)^n\beta^{-n} - \beta^n]$, we see that F_n cannot be expressed as a polynomial in β^n (nor, indeed, is β greater than 1, must less an integer). Therefore, the equation

(12) $\quad F_n = p(m_n)$, where $\deg(p) \geq 2$,

necessarily has only a finite number of solutions, for all acceptable given polynomials p.

The conditions sought for part B of the problem are those imposed by the conditions of the Theorem. Unless $f_n = p(z_1^n)$ for all n, where z_1 is an integer greater than 1, the equation $f_n = p(m_n)$ must have a finite number of solutions.

Note that the conditions of the Theorem are satisfied by the example cited in part B, with $m_n = 2^n$, $z_1 = 2$, $p(z) = z^2 + z$.

True or Not?

H-457 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***
(Vol. 29, no. 3, August 1991)

Let $f(N)$ denote the number of addends in the Zeckendorf decomposition of N. The numerical evidence resulting from a computer experiment suggests the following two conjectures. Can they be proved?

Conjecture 1: For given positive integers k and n, there exists a positive integer n_k (depending on k) such that $f(kF_n)$ has a constant value for $n \geq n_k$.

For example,

$$24F_n = F_{n+6} + F_{n+3} + F_{n+1} + F_{n-4} + F_{n-6} \text{ for } n \geq 8.$$

By inspection, we see that $n_1 = 1$, $n_k = 2$ for $k = 2$ or 3, $n_4 = 4$ and $n_k = 5$ for $5 \leq k \leq 8$.

Conjecture 2: For $k \geq 6$, let us define (i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k \leq L_\mu$, and (ii) ν, the subscript of the largest Fibonacci number such that $k > F_\nu + F_{\nu-6}$. Then, $n_k = \max(\mu, \nu)$.

Solution by Paul S. Bruckman, Edmonds, WA

We suppose $n \geq 2$. As we know, any natural number u has a unique Zeckendorf representation (Z-rep. for short) which is given by:

(1) $\quad u = \sum_{j=2}^{r} \theta_j F_j$, where $\theta_j = 0$ or 1, $\theta_j\theta_{j+1} = 0$, $j = 2, 3, \ldots, r-1$, and $\theta_r = 1$.

We shall show that Conjecture 1 is true, Conjecture 2 false. Moreover, the following "observations" are the correct ones for n_k: $n_1 = 2$, $n_k = 4$ for $2 \leq k \leq 4$, $n_k = 6$ for $5 \leq k \leq 11$, $n_k = 8$ for $12 \leq k \leq 29$, etc.; in general:

(2) $\quad n_k = 2m + 2$, where m is determined by $L_{2m-1} < k \leq L_{2m+1}$, $m = 1, 2, \ldots$.

Therefore,

(3) $\quad n_k = 1 + \mu$, where μ is as defined by the proposer.

To prove the assertions in (2) and (3), it will suffice to prove (4) and (5) below.

(4) Given k such that $L_{2m-1} < k < L_{2m}$, then for all $n \ge 2m+2$ there exists a Z-rep. for kF_n given by:

$$kF_n = \sum_{j=-2m}^{2m-1} \theta_j^{(k)} F_{n+j}, \text{ where } \theta_{-2m}^{(k)} = \theta_{2m-1}^{(k)} = 1.$$

(5) Given k such that $L_{2m} \le k \le L_{2m+1}$, then for all $n \ge 2m+2$ there exists a Z-rep. for kF_n given by:

$$kF_n = \sum_{j=-2m}^{2m} \theta_j^{(k)} F_{n+j}, \text{ where } \theta_{-2m}^{(k)} = \theta_{2m}^{(k)} = 1.$$

In these expressions, the $\theta_j^{(k)}$'s are dependent on k but not on n. In the sequel, we shall frequently employ sums of the type

$$\sum_{j=r}^{s} \theta_j^{(k)} F_{n+j}.$$

For brevity, we shall denote such a sum by $S(r, s)$, If we wish to emphasize that $\theta_s^{(k)} = 1$, we shall use the notation $S(r, \underline{s})$; similar notation makes the symbols $S(\underline{r}, s)$ and $S(\underline{r}, \underline{s})$ self-explanatory. Of course, all such sums are understood to be Z-reps. Some preliminary lemmas are needed to prove (4) and (5).

Lemma 1:

(6) (i) $2F_n = F_{n+1} + F_{n-2}$; (ii) $3F_n = F_{n+2} + F_{n-2}$; (iii) $4F_n = F_{n+2} + F_n + F_{n-2}$.

We omit the proof, as this is readily verified. Note that the right member of the expressions in (i)-(iii) are Z-reps., with r = -2, and are therefore valid for all $n \ge 4$. Since $f(kF_n) = 2$, $k = 2, 3$, and $f(4F_n) = 3$ for all $n \ge 4$, it follows that $n_k = 4$ for $k = 2, 3, 4$. Of course, $F_n = F_n$ for all $n \ge 2$, so $n_1 = 2$.

Lemma 2:

(7) $L_{2m}F_n = F_{n+2m} + F_{n-2m}$.

This is also readily verified. Note that the right member of (7) is of the form $S(-\underline{2m}, \underline{2m})$, and is in fact the *unique* $S(-\underline{2m}, \underline{2m})$ of minimum length. Thus, $f(L_{2m}F_n) = 2$ for all $n \ge 2m+2$; hence, $n_{L_{2m}} = 2m+2$.

Lemma 3:

(8) $L_{2m+1}F_n = \sum_{j=-m}^{m} F_{n+2j} = F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m}$.

We omit the proof, leaving it as an exercise. Note that $L_{2m+1} = L_{2m+2} - L_{2m}$, which leads to the second relation in (8), using Lemma 2. The sum in (8) is a Z-rep. of the form $S(-\underline{2m}, \underline{2m})$, valid for all $n \ge 2m+2$. Hence, $f(L_{2m+1}F_n) = 2m+1$ for all $n \ge 2m+2$, and $n_{L_{2m+1}} = 2m+2$.

We now proceed to the proof of (4) and (5), by induction on m. Let T denote the set of all positive integers m for which (4) and (5) are both true. (4) is true for $m=1$ $(k=2)$, and (5) is true for $m=1$ $(k=3,4)$, by Lemma 1. Therefore, $1 \in T$. Suppose $1, 2, \ldots, m \in T$ (the inductive hypothesis). We break up our proof into six subcases:

Case 1. Suppose $5F_{2m} < k < L_{2m+2}$. Then $L_{2m-1} < k - L_{2m+1} < L_{2m}$. Using (4) (supposed true for m), we have:

$$(k-L_{2m+1})F_n = \sum_{j=-2m}^{2m-1} \theta_j^{(k)} F_{n+j} \text{ for all } n \geq 2m+2.$$

Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+2}, \underline{2m-1}) + F_{n+2m+1} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+1}), \text{ for all } n \geq 2m+4, \end{aligned}$$

which is the statement of (4) for $m+1$.

Case 2. Suppose $2L_{2m} \leq k \leq 5F_{2m}$. Then $L_{2m-2} \leq k - L_{2m+1} \leq L_{2m-1}$. Using (5) for $m-1$,

$$(k-L_{2m+1})F_n = S(-\underline{2m+2}, \underline{2m-2}) \text{ for all } n \geq 2m.$$

Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m+2}, \underline{2m-2}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+1}, \underline{2m-2}) + F_{n+2m+1} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+1}), \text{ for all } n \geq 2m+4, \end{aligned}$$

which is the statement of (4) for $m+1$.

Case 3. Suppose $L_{2m+1} < k < 2L_{2m}$. Then $L_{2m-1} < k - L_{2m} < L_{2m}$. By (4), for m,

$$(k-L_{2m})F_n = S(-\underline{2m}, \underline{2m-1}) \text{ for all } n \geq 2m+2.$$

Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m} + F_{n-2m} \\ &= S(-2m+2, 2m-3) + 2F_{n-2m} + F_{n+2m-1} + F_{n+2m} \\ &= S(-2m+2, 2m-3) + F_{n-2m+1} + F_{n-2m-2} + F_{n+2m+1}. \end{aligned}$$

If $\theta_{-2m+2}^{(k)} = 0$, then

$$kF_n = S(-\underline{2m+1}, \underline{2m+1}) + F_{n-2m-2} = S(-\underline{2m-2}, \underline{2m+1}).$$

If $\theta_{-2m+2}^{(k)} = 1$, then $\theta_{-2m+3}^{(k)} = 0$, and

$$\begin{aligned} kF_n &= S(-2m+4, 2m-3) + F_{n-2m+2} + F_{n-2m+1} + F_{n-2m-2} + F_{n+2m+1} \\ &= S(-2m+4, \underline{2m+1}) + F_{n-2m-2} + F_{n-2m+3}. \end{aligned}$$

Since $\theta_{2m+1}^{(k)} = \theta_{2m-3}^{(k)} = 1$, we must have $\theta_{2j}^{(k)} = 0$ for at least one j with $-m+2 \le j \le m-3$, and certainly $\theta_{2m-4}^{(k)} = \theta_{2m-2}^{(k)} = \theta_{2m}^{(k)} = 0$. Thus, $kF_n = S(-\underline{2m+2r+1}, \underline{2m+1}) + F_{n-2m-2}$ for some $r \ge 0$, which implies $kF_n = S(-\underline{2m-2}, \underline{2m+1})$ for all $n \ge 2m+4$. This is the statement of (4) for $m+1$.

Combining cases 1, 2, and 3, we see that if $L_{2m+1} < k < L_{2m+2}$, then the assertion of (4) for $m+1$ is valid. Thus, $m \in T$ implies (4) for $m+1$.

Case 4. Suppose $L_{2m+2} \le k \le 2L_{2m+1}$. Then $L_{2m} \le k - L_{2m+1} \le L_{2m+1}$. By (5), for m, $(k - L_{2m+1})F_n = S(-\underline{2m}, \underline{2m})$ for all $n \ge 2m+2$. Then, by Lemma 3,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m}) + F_{n+2m+2} + F_{n-2m-2} - F_{n+2m} - F_{n-2m} \\ &= S(-\underline{2m+2}, \underline{2m-2}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4, \end{aligned}$$

which is the statement of (5) for $m+1$.

Case 5. Suppose $2L_{2m+1} < k < 5F_{2m+1}$. Then $L_{2m-1} < k - L_{2m+2} < L_{2m}$. By (4), for m, $(k - L_{2m+2})F_n = S(-\underline{2m}, \underline{2m-1})$ for all $n \ge 2m+2$. Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m-1}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4, \end{aligned}$$

which is the statement of (5) for $m+1$.

Case 6. Suppose $5F_{2m+1} \le k \le L_{2m+3}$. Then $L_{2m} \le k - L_{2m+2} \le L_{2m+1}$. Then, using (5), for m, $(k - L_{2m+2})F_n = S(-\underline{2m}, \underline{2m})$ for all $n \ge 2m+2$. Then, by Lemma 2,

$$\begin{aligned} kF_n &= S(-\underline{2m}, \underline{2m}) + F_{n+2m+2} + F_{n-2m-2} \\ &= S(-\underline{2m-2}, \underline{2m+2}) \text{ for all } n \ge 2m+4. \end{aligned}$$

This is the statement of (5) for $m+1$.

Combining cases 4, 5, and 6, we see that if $L_{2m+2} \le k \le L_{2m+3}$ and $m \in T$ is assumed, then (5) holds for $m+1$. Combining this conclusion with the conclusion of case 3, we see that $m \in T$ implies $(m+1) \in T$. Since $1 \in T$, the proof of (4) and (5) by induction is complete.

These relations, in turn, imply the truth of the original assertions [(2) and (3)]. For (4) and (5) they may be combined as follows:

(9) Given k such that $L_{2m-1} < k \le L_{2m+1}$, then for all $n \ge 2m+2$,

$kF_n = S(-\underline{2m}, 2m)$, and $\theta_{2m}^{(k)} + \theta_{2m-1}^{(k)} = 1$.

We see from (9) that $n_k = 2m+2$, where $2m+1 = \mu$, as defined by the proposer. This proves (3). Q.E.D.

Editorial Note: Russell Hendel's name was omitted from the list of solvers of H-453.

❖❖❖