CYCLIC BINARY STRINGS WITHOUT LONG RUNS OF LIKE (ALTERNATING) BITS

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1. A binary n-bit *cyclic* string (briefly n-CS) is a sequence of n 0's and 1's (the bits), with the first and last bits considered to be adjacent (i.e., the first bit follows the last bit). This condition is visible when the string is displayed in a circle with one bit "capped": the capped bit is the first bit and reading clockwise we see the second bit, the third bit, and so on to the n^{th} bit (the last bit). In an n-CS, a subsequence of consecutive bits is a *run*. Motivated by a problem of genetic information processing, Agur, Fraenkel, and Klein [1] derived formulas for the number of n-CSs with no runs 0 0 0 nor 1 1 1 (i.e., all runs of like bits have length ≤ 2) and for the number with no runs 0 1 0 nor 1 0 1 (i.e., all runs of alternating bits have length ≤ 2). These are the cases $w = 2$ of

$$L_{\leq w}(n) = \text{the number of } n\text{-CSs in which all runs of like bits have length} \leq w$$

and

$$A_{\leq w}(n) = \text{the number of } n\text{-CSs in which all runs of alternating bits have length} \leq w.$$

In this note we prove the

Theorem:

$$L_{\leq w}(n) = \begin{cases} 2^n, & \text{if } 1 \leq n \leq w-1, \\ F_w(n) + D_w(n), & \text{if } n \geq w, \end{cases}$$

and

$$A_{\leq w}(n) = \begin{cases} 2^n, & \text{if } 1 \leq n \leq w-1, \\ F_w(n) + (-1)^n D_w(n), & \text{if } n \geq w, \end{cases}$$

where

(1)
$$F_w(0) = w, \quad F_w(n) = 2^n - 1, \quad 1 \leq n \leq w-1,$$
$$F_w(n) = F_w(n-1) + F_w(n-2) + \cdots + F_w(n-w), \quad n \geq w,$$

and

(2)
$$D_w(n) = \begin{cases} w, & \text{if } n \geq 1 \text{ and } w+1 \mid n, \\ -1, & \text{if } n \geq 1 \text{ and } w+1 \nmid n. \end{cases}$$

Furthermore,

(3)
$$L_{\leq w}(n) \sim A_{\leq w}(n) \sim c\alpha^n$$

where c is a constant (which depends only on w) and

$$2 - \frac{2}{2^w} < \alpha < 2 - \frac{1}{2^w}.$$

2. Consider for any n-CS

$$x = x_1x_2x_3\ldots x_n,\ \ x_i = 0, 1,$$

the n-CS

$$T(x) = y_1y_2\ldots y_n, \quad y_i = \begin{cases}0, & \text{if } x_i = x_{i-1},\\ 1, & \text{if } x_i \neq x_{i-1},\end{cases} \ i = 1, 2, \ldots, n \ \ (x_0 = x_n).$$

For example,

$$T(0011100101100011111) = 1010010111010010000$$
$$T(1011000111010101111) = 0110100100111111100$$
$$T(1100110011001000000) = 1010101010101110000$$

Thus, when passing over the bits of x, $T(x)$ records the changes (from 0 to 1 or from 1 to 0) by a 1, and records no change (from 0 to 0 or from 1 to 1) by a 0.

Of course

$$T(\tilde{x}) = T(x),$$

where $\tilde{x}$ is the complementary n-CS

$$\tilde{x} = y_1y_2y_3\ldots y_n, \quad y_i = \begin{cases}1, & \text{if } x_i = 0,\\ 0, & \text{if } x_i = 1.\end{cases}$$

However, for any two <u>different</u> n-CSs u and v, <u>both with first bit 1</u>, $T(u) \neq T(v)$. Indeed, T is bijective between the set of 2^{n-1} n-CSs with first bit 1 and the set of 2^{n-1} n-CSs with an even number of 1's. Thus, an n-CS x with first bit 1 corresponds to an n-CS $T(x)$ with an even number of 1's, and then a run of w like bits in x corresponds to a run of $w-1$ 0's in $T(x)$, while a run of w alternating bits in x corresponds to a run of $w-1$ 1's in $T(x)$.

Hence,

x is an n-CS with first bit 1 and all runs of like bits have length $\leq w$
if and only if
$T(x)$ is an n-CS with an even number of 1's and all runs of 0's have length $\leq w-1$,

so we have

$$L_{\leq w}(n) = 2B^e_{w-1}(n),\ \ n \geq 1,$$

where

$B^e_w(n)$ = the number of n-CSs with an even number of 1's and all runs of 0's have length $\leq w$.

Also,

x is an n-CS with first bit 1 and all alternating runs have length $\leq w$
if and only if
$T(x)$ is an n-CS with an even number of 1's and all runs of 1's have length $\leq w-1$
if and only if
$\widetilde{T(x)}$ is an n-CS with an even number of 0's and all runs of 0's have length $\leq w-1$
if and only if

n is even, $\widetilde{T(x)}$ has an even number of 1's and
all runs of 0's have length $\leq w-1$
or
n is odd, $\widetilde{T(x)}$ has an odd number of 1's and
all runs of 0's have length $\leq w-1$,

and we have

$$A_{\leq w}(n)=\begin{cases}2B^e_{w-1}(n), & \text{if } n \text{ is even,}\\ 2B^o_{w-1}(n), & \text{if } n \text{ is odd,}\end{cases}$$

where

$B^o_w(n)$ = the number of n-CSs with an odd number of 1's and all runs of 0's have length $\leq w$.

In terms of $B_w(n)=B^e_w(n)+B^o_w(n)$ and $C_w(n)=B^e_w(n)-B^o_w(n)$,

(4) $$L_{\leq w}(n)=B_{w-1}(n)+C_{w-1}(n),\quad A_{\leq w}(n)=B_{w-1}(n)+(-1)^n C_{w-1}(n),\quad n\geq 1.$$

In order to determine $B^e_w(n)$, $B^o_w(n)$, and $B_w(n)$, we naturally investigate (for $n\geq 1$, $k\geq 0$, $w\geq 1$) $(n:k)_w$ = the number of n-CSs with exactly k 1's ($n-k$ 0's) and all runs of 0's have length $\leq w$. Clearly

(5) $$(n:0)_w=\begin{cases}1, & 1\leq n\leq w,\\ 0, & n\geq w+1,\end{cases}$$

and

(6) $$(n:k)_w=\begin{cases}\binom{n}{k}, & 1\leq n\leq w+k,\\ 0, & 1\leq k,\ k(w+1)<n,\end{cases}$$

where

$$\binom{n}{k}=\begin{cases}n!/k!(n-k)!, & 0\leq k\leq n,\\ 0, & 0\leq n<k.\end{cases}$$

Consider an n-CS counted in $(n:k)_w$, $n\geq w+2$, $k\geq 2$. If the first bit is 1 (i.e., capped bit is $\hat{1}$), and the last 1 is followed by exactly i 0's ($0\leq i\leq w$), delete this last 1 and the i 0's which follow it and then we have an $(n-1-i)$-CS with $k-1$ 1's, first bit 1, and every run of 0's has length $\leq w$. If the first bit is 0, and the first 1 is followed by i 0's ($0\leq i\leq w$), delete this first 1 and the i 0's which follow it and we then have an $(n-1-i)$-CS with $k-1$ 1's, first bit 0, and every run of 0's has length $\leq w$. Hence,

(7) $$(n:k)_w=(n-1:k-1)_w+(n-2:k-1)_w+\cdots+(n-1-w:k-1)_w,\ k\geq 2,\ n\geq w+2.$$

Of course,

$$B^e_w(n)=\sum_{k=0}(n:2k)_w,\quad B^o_w(n)=\sum_{k=1}(n:2k-1)_w.$$

From (5), (6), and (7), we deduce that

$$B_w^e(n)=\begin{cases}2^{n-1}, & 1\le n\le w,\\ 2^{n-1}-1, & n=w+1,\\ B_w^o(n-1)+B_w^o(n-2)+\cdots+B_w^o(n-1-w), & n\ge w+2,\end{cases}$$

$$B_w^o(n)=\begin{cases}2^{n-1}, & 1\le n\le w+1,\\ B_w^e(n-1)+\cdots+B_w^e(n-1-w)+n-2(w+1), & w+2\le n\le 2w+1,\\ B_w^e(n-1)+B_w^e(n-2)+\cdots+B_w^e(n-1-w), & n\ge 2w+2,\end{cases}$$

$$B_w(n)=B_w^e(n)+B_w^o(n)$$

$$=\begin{cases}2^n, & n=1,2,\ldots,w,\\ 2^n-1, & n=w+1,\\ B_w(n-1)+\cdots+B_w(n-1-w)+n-2(w+1), & w+2\le n\le 2w+1,\\ B_w(n-1)+B_w(n-2)+\cdots+B_w(n-1-w), & n\ge 2w+2.\end{cases}$$

Furthermore, the numbers $C_w(n)=B_w^e(n)-B_w^o(n)$ are seen to satisfy

$$C_w(n)=\begin{cases}0, & 1\le n\le w,\\ -1, & n=w+1,\\ w+1, & n=w+2,\\ -1, & w+3\le n\le 2w+1,\\ -\{C_w(n-1)+C_w(n-2)+\cdots+C_w(n-1-w)\}, & n\ge 2w+2,\end{cases}$$

that is,

$$C_w(n)=\begin{cases}0, & 1\le n\le w,\\ w+1, & n\ge w+1,\ w+2\mid n,\\ -1, & n\ge w+1,\ w+2\nmid n.\end{cases}$$

Now it is easy to verify that

$$B_{w-1}(n)=F_w(n),\ n\ge w,\quad C_{w-1}(n)=D_w(n),\ n\ge w,$$

where $F_w(n)$ and $D_w(n)$ are defined by (1) and (2), and this with (4) completes the first part of the Theorem.

It is well known that any sequence $\{H_w(n)\}_{n=0}$ which satisfies

$$H_w(n)=H_w(n-1)+H_w(n-2)+\cdots+H_w(n-w),\ n\ge w,$$

can be written

$$H_w(n)=\sum_{i=1} c_i\alpha_i^n,\ n=0,1,2,\ldots,$$

where $\alpha_i=\alpha_i^{(w)},\ (i=1,2,\ldots,w)$ are the roots of

(8) $$z^w-z^{w-1}-z^{w-2}-\cdots-z-1$$

and the $c_i=c_i^{(w)},\ (i=1,2,\ldots,w)$ are determined by the w equations

$$c_1\alpha_1^n+c_2\alpha_2^n+\cdots+c_w\alpha_w^n=H_w(n),\ n=0,1,\ldots,w-1.$$

This depends on the fact that $\alpha_1, \alpha_2, \ldots, \alpha_w$ are distinct. This fact is easily proved: multiply (8) by $x-1$ to get $x^{w+1}-2x^w+1$ which has no multiple roots because it has no roots in common with its derivative.

Cappocelli and Cull [2] have shown that exactly one of the roots of (8) is real and positive, say $\alpha = \alpha_1$, and it satisfies

$$(9) \qquad 2-\frac{2}{2^w} < \alpha < 2-\frac{1}{2^w},$$

while all the other roots satisfy

$$\frac{1}{\sqrt[w]{3}} < |\alpha_i| < 1, \quad i = 2,3,\ldots,w.$$

It follows that $H_w(n) \sim c\alpha^n$. This leads to

$$L_{\le 2}(n) \sim A_{\le 2}(n) \sim \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

and for $w \ge 2$, $L_{\le w}(n) \sim A_{\le w}(n) \sim c\alpha^n$, where c is a constant (which depends only on w) and α satisfies (9).

The following tables show $F_w(n)$, $D_w(n)$, $L_{\le w}(n)$, and $A_{\le w}(n)$ for $w = 2, 3, 4$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843
$D_2(n)$		−1	−1	2	−1	−1	2	−1	−1	2	−1	−1	2	−1	−1
$L_{\le 2}(n)$		2	2	6	6	10	20	28	46	78	122	198	324	520	842
$A_{\le 2}(n)$		2	4	2	6	12	20	30	46	74	122	200	324	522	842

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_3(n)$	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757
$D_3(n)$		−1	−1	−1	3	−1	−1	−1	3	−1	−1	−1	3	−1
$L_{\le 3}(n)$		2	4	6	14	20	38	70	134	240	442	814	1502	2756
$A_{\le 3}(n)$		2	4	8	14	22	38	72	134	242	442	816	1502	2578

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_4(n)$	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071
$D_4(n)$		−1	−1	−1	−1	4	−1	−1	−1	−1	4	−1	−1	−1
$L_{\le 4}(n)$		2	4	8	14	30	50	98	190	366	712	1364	2630	5070
$A_{\le 4}(n)$		2	4	8	14	22	50	100	190	368	712	1366	2630	5072

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