FIBONACCI-TYPE SEQUENCES AND MINIMAL SOLUTIONS OF DISCRETE SILVERMAN GAMES

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1. GAME THEORY BACKGROUND

While the principal results of this paper seem to us to be of interest in their own right, and can be understood with no reference to game theory, the problems addressed arose in a game theory setting, and their solution has important consequences for the analysis of Silverman games. It seems appropriate therefore to sketch briefly the game theory background. Silverman games are two-person, zero-sum games in which, roughly speaking, the higher bid wins, unless it is too much higher than the other, in which case it loses. More precisely, let S_{I} and S_{II} be sets of positive real numbers, and T and v be parameters with $T > 1$ and $v > 0$. The sets S_{I} and S_{II} are the pure strategy sets for Players I and II, respectively. Each player chooses a number from his strategy set, and the higher number wins 1, unless it is at least T times as large as the other, in which case it loses v. The parameters T and v are referred to as the threshold and the penalty, respectively. If $S_{\text{I}} = S_{\text{II}}$, the game is symmetric, and in this case, if optimal strategies exist they are the same for both players, and the game value is 0.

The prototype games are attributed to David Silverman, although the earliest published mention of such a game of which we are aware is by Herstein and Kaplansky ([3], p. 212). The symmetric game on an open interval was analyzed by R. J. Evans [1] for arbitrary T and v, and the symmetric game on discrete sets by Evans and Heuer [2]. An analogous symmetric game on $[1, \infty)$ is examined in [5]. Discrete games with $S_{\text{I}} \cap S_{\text{II}} = \varnothing$ are examined in [4] and [8]. In [6] it is shown that when $v \geq 1$ Silverman games reduce by dominance to games on bounded sets, and in [7] this and other types of dominance are used to reduce discrete games with $v \geq 1$ to finite games, and their payoff matrices have a simple characteristic form.

Many semi-reduced games can be further reduced in the sense that there still are proper subsets W_{I} and W_{II} of the strategy sets, with the property that optimal mixed strategies for the game on $W_{\text{I}} \times W_{\text{II}}$ are optimal for the full game. This further reduction leads to games some of which are 2×2 and the rest of which fall into eight families, four of even-order games and four of odd-order games (see [7]). It was our conjecture that when $v > 1$, no further reduction of any of these games is possible. This would mean that optimal mixed strategies for such a reduced game are minimal optimal strategies for the original game. We shall show here that, for the odd-order games, this is indeed the case, and using similar techniques we obtain explicitly the unique optimal mixed strategies and game values for these reduced games. The even-order cases will be treated in a forthcoming paper.

2. THE ASSOCIATED MATRICES

Let B denote the payoff matrix of our reduced game and V the game value. Then B is always square, and as discussed in Section 13 of [7], the game is not further reducible if and only if there is a unique probability vector P, with all components positive, such that

$$PB = (V, V, \dots, V). \tag{2.1a}$$

In this case there is also a unique probability vector Q such that

$$BQ^t = (V, V, \dots, V)^t, \tag{2.1b}$$

and P and Q are the unique optimal mixed strategy vectors for the row player and column player, respectively. (We are writing vectors as row vectors.)

Let $B_{\cdot j}$ denote the j^{th} column of B. If B is $2n+1$ by $2n+1$, then (2.1a) is equivalent to

$$PB_{\cdot j} = V \quad \text{for } j = 1, 2, \dots, 2n+1. \tag{2.2}$$

With the understanding that P is to be a probability vector, this, in turn, is equivalent to

$$P(B_{\cdot j} - B_{\cdot j+1}) = 0 \quad \text{for } j = 1, 2, \dots, 2n, \quad \text{and} \quad \sum_{i=1}^{2n+1} p_i = 1, \text{ with each } p_i > 0. \tag{2.3}$$

Now let A be the $2n+1$ by $2n+1$ matrix, the i^{th} row of which is $(B_{\cdot i} - B_{\cdot i+1})^t$ for $i = 1, 2, \dots, 2n$, and the $(2n+1)^{\text{th}}$ row of which is $(1, 1, \dots, 1)$. Then (2.3) is equivalent to

$$AP^t = (0, 0, \dots, 0, 1)^t, \tag{2.4}$$

which has a unique solution if and only if A is nonsingular. Thus, it suffices to show that A is nonsingular and that a probability vector P with all components positive exists, satisfying (2.4).

The four families of odd-order payoff matrices B and the associated matrices A are illustrated below. The variable x is $1 + v$, and with $v > 1$ we have $x > 2$. Types (i), (ii), (iii), and (iv) here correspond to (8.0.5A), (8.0.5B), (8.0.5C), and (8.0.5D), respectively, in [7]. The main diagonal and first superdiagonal of A consist entirely of 1s, with two exceptions. In column $a + 1$, the pair $\binom{2}{0}$ occurs in place of $\binom{1}{1}$, and in column $n + a + 2$, $\binom{0}{2}$ occurs. In general, the matrix A of type (i) has a columns preceding the first irregular one, then d regular columns, a central column, a regular columns, the second irregular one, and d regular ones, for a total of $2n+1 = 2a+2d+3$ columns.

$$B = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\ 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v \\ 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\ -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ -v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\ -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Type (i), parameters $a \geq 0, d \geq 0; n = a+d+1$. Illustrated with $a = 2, d = 2$.

In the matrix A of type (ii), there are three irregular columns. The parameters here are c and d, and the pattern is $c + 1$ regular columns, the column with the $\binom{2}{0}$, d regular columns, the central column, c regular columns, two columns with $\binom{0}{2}$ in place of $\binom{1}{1}$, and d regular columns. We illustrate it here with $c = 1, d = 2; n = c+d+2 = 5$, so again B and A are 11×11.

$$B=\begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\ 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v \\ 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\ -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ -v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix};$$

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\ -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Type (ii), parameters $c \geq 0, d \geq 0; n = c+d+2$. Illustrated with $c = 1, d = 2$.

We illustrate type (iii) below.

$$B=\begin{pmatrix}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix};$$

$$A=\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\
-x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

Type (iii), parameters $a \geq 0, b \geq 0; n = a+b+2$. Illustrated with $a = 2, b = 1$.

In the matrix A of type (iii), shown above, there are again three irregular columns. The parameters are a and b, and the pattern of columns is: a regular columns, two columns with $\binom{2}{0}$ in place of $\binom{1}{1}$, b regular columns, the central column, a regular, one with $\binom{0}{2}$ and $b+1$ regular.

Finally, in matrix A of type (iv), there are two irregular columns. The parameters are denoted c and b, and the pattern of columns is $c+1$ regular, one with $\binom{2}{0}$, b regular, the central column, c regular columns, one with $\binom{0}{2}$, and $b+1$ regular. We illustrate type (iv) below, with $c = 2, b = 1$; $n = c+b+2 = 5$.

$$B=\begin{pmatrix}
0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v & v \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v & v \\
1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v & v & v \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & v & v \\
1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & v \\
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\
-v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
-v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\
-v & -v & -v & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\
-v & -v & -v & -v & -v & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix};$$

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -x \\ -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Type (iv), parameters $c \ge 0, b \ge 0; n = c+b+2$. Illustrated with $c = 2, b = 1$.

Main Theorem: For $x > 2$, every matrix in these four two-parameter families is nonsingular, and the unique vector P satisfying (2.4) has all components positive.

When the diagonal of the payoff matrix B consists entirely of zeros the game is symmetric, and has been shown in [2] to have a unique optimal mixed strategy. It follows in that case that the associated matrix of (2.4), which we denote A^*, is nonsingular. This matrix A^* is like those in the four families above, but without the irregularities; i.e., the main diagonal and the first super-diagonal consist entirely of 1s. We shall in each instance prove that A is nonsingular by exhibiting a matrix D such that $AD = A^*$, and prove that a completely mixed (all components positive) vector P satisfying (2.4) exists by exhibiting it. The task of obtaining such a D is lightened substantially by the observation that in each of the four classes, the matrix A differs from A^* in at most two columns. It suffices, therefore, to show that these columns of A^* lie in the column space of A, and we accomplish this by producing columns $D_{\cdot j}$ such that $AD_{\cdot j} = A^*_{\cdot j}$ for the appropriate j.

We illustrate here using the case $a = d = 1$ in type (i). Then $n = 3$, and the matrices A and A^* are 7×7.

$$A=\begin{pmatrix} 1 & 2 & 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -x & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -x \\ -x & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -x & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This matrix differs from A^* only in columns 2 $(= a+1)$ and 6 $(= n+a+2)$. The column $D_{\cdot a+1}$ is given by (4.0.2), and in this illustration it is

$$D_{\cdot 2}=\begin{pmatrix} -2(x+2)T_0 + x(x+2)T_1 + 2 \\ x(x+2)E_2 + 1 \\ -2x(x+2)E_1 + x^2(x+2)E_{-1} + x \\ -2x(x+2)E_0 + x^2(x+2)E_0 + x \\ -2x(x+2)E_{-1} + x^2(x+2)E_1 + x \\ -(x+2)T_2 + 1 \\ -2(x+2)T_1 + x(x+2)T_0 + 2 \end{pmatrix}.$$

If $\Delta = x(x+2)R_2$ [see (4.0.1)], the reader may verify, using identities (3.2), (3.0.5), (3.0.3), (3.8), and (3.9), and particular values of E_n and T_n given by (3.0.10) and (3.0.13), that $AD_2 = \Delta A_2^*$, where $A_2^* = (1, 1, 0, 0, -x, 0, 1)^t$ (This is then a special case of Theorem 4.1.)

3. THE POLYNOMIAL SEQUENCES

We shall describe the matrix D in terms of six Fibonacci-like sequences of polynomials, and use Fibonacci-like properties of these sequences to prove that $AD = A^*$. Each sequence is a particular solution to the recursion

$$Y_{m+1} = (x^2-2)Y_m - Y_{m-1} + C, \tag{3.0.1}$$

where the constant C is 0, 1, or 2. For some earlier work on sequences generated by a recursion like (3.0.1) without the (x^2-2) coefficient (see [9] and [10]).

Define polynomial sequences E_m, R_m, G_m, T_m, H_m, and K_m as follows:

$$E_0 = 1, E_1 = x^2-1, E_{m+1} = (x^2-2)E_m - E_{m-1} + 1. \tag{3.0.2}$$

$$R_m = E_m - E_{m-1}. \tag{3.0.3}$$

$$G_m = R_m - R_{m-1}. \tag{3.0.4}$$

$$T_m = E_m + E_{m-1}. \tag{3.0.5}$$

$$H_m = R_m + R_{m-1} = E_m - E_{m-2} = T_m - T_{m-1}. \tag{3.0.6}$$

$$K_m = H_m - H_{m-1} = R_m - R_{m-2} = G_m + G_{m-1} = T_m - 2T_{m-1} + T_{m-2}. \tag{3.0.7}$$

In (3.0.6) and (3.0.7) the first equality is to be understood as the definition; the others follow immediately. One sees further at once that

$$R_m, G_m, H_m, \text{ and } K_m \text{ satisfy (3.0.1) with } C = 0, \tag{3.0.8}$$

and that

$$T_m \text{ satisfies (3.0.1) with } C = 2. \tag{3.0.9}$$

The recursion (3.0.1) can be used to extend the sequence in both directions, and we regard each of the sequences as being defined for all integers m. From the recursions, one finds readily the following:

$$E_{-1} = E_{-2} = 0,\ E_{-3} = E_0 - 1, \text{ and } E_{-m} = E_{m-3}. \tag{3.0.10}$$

$$R_0 = 1,\ R_{-1} = 0,\ R_{-2} = -1, \text{ and } R_{-m} = -R_{m-2}. \tag{3.0.11}$$

$$G_0 = G_{-1} = 1, \text{ and } G_{-m} = G_{m-1}. \tag{3.0.12}$$

$$T_0 = 1,\ T_{-1} = 0,\ T_{-2} = 1, \text{ and } T_{-m} = T_{m-2}. \tag{3.0.13}$$

$$H_0 = 1,\ H_{-1} = -1, \text{ and } H_{-m} = -H_{m-1}. \tag{3.0.14}$$

$$K_1 = x^2-2,\ K_0 = 2, \text{ and } K_{-m} = K_m. \tag{3.0.15}$$

Theorem 3.1: Every polynomial E_m with $m \geq 0$ takes only positive values for $x > 2$. The same is true of each of the other sequences defined by (3.0.2) to (3.0.7).

Proof: It is a routine exercise to prove by induction that $E_{m+1} \geq E_m \geq 0$ for $x > 2$ and all m. The same goes for each of the other sequences.

Following are some further properties of these polynomials that we will find useful.

$$x^2 E_m = T_m + T_{m+1} - 1. \tag{3.2}$$

This is immediate from (3.0.2) and (3.0.5).

Similarly, from the recursion (3.0.8) for G_m and (3.0.7), we have

$$x^2 G_m = K_{m+1} + K_m, \tag{3.3}$$

and from the recursion (3.0.8) for R_m and (3.0.6),

$$x^2 R_m = H_{m+1} + H_m. \tag{3.4}$$

From (3.0.8) and (3.0.4) we obtain

$$(x^2 - 4)R_m = G_{m+1} - G_m, \tag{3.5}$$

and from (3.0.2), (3.0.3), and (3.0.4), we have

$$(x^2 - 4)E_m + 1 = G_{m+1}. \tag{3.6}$$

Similarly we obtain

$$(x^2 - 4)T_m + 2 = K_{m+1}. \tag{3.7}$$

From (3.0.9) we have that $(x^2 - 2)T_i - T_{i+1} - T_{i-1} = -2$. Upon summing this for $0 \leq i \leq m$, adding $T_{m+1} - T_m - 1$ to both sides, and using (3.0.13), we obtain

$$(x^2 - 4)\sum_{i=0}^{m} T_i = T_{m+1} - T_m - 2m - 3. \tag{3.8}$$

In exactly the same way, using (3.0.2) and (3.0.10), we obtain

$$(x^2 - 4)\sum_{i=0}^{m} E_i = E_{m+1} - E_m - m - 2. \tag{3.9}$$

Theorem 3.10: For all integers r and m,

$$G_r H_m + G_m H_r = 2R_{r+m}. \tag{3.10.1}$$

Proof: For fixed r, both members are sequences indexed by m satisfying the homogeneous difference equation (3.0.1), as noted in (3.0.8). It will suffice, therefore, to show equality in (3.10.1) for $m = -1$ and $m = 0$. But from (3.0.4) and (3.0.6) we have $-G_r + H_r = 2R_{r-1}$ and $G_r + H_r = 2R_r$, which, in view of (3.0.12) and (3.0.14), establishes (3.10.1) for $m = -1$ and $m = 0$.

Theorem 3.11: For all integers r and m,

$$G_r R_m + G_m R_{r-1} = R_{r+m}. \tag{3.11.1}$$

Proof: This is proved in the same way as (3.10), using (3.0.4).

In much the same way, one shows

$$G_r R_m - G_{r-1} R_{m-1} = G_{r+m}, \tag{3.12}$$

$$R_r H_m - R_{m-1} H_r = R_{r+m}, \tag{3.13}$$

$$K_{r+1} R_m - G_r H_m = G_{r+m+1}, \tag{3.14}$$

$$K_{r+1} R_m + K_{m+1} R_r = 2R_{r+m+1}, \tag{3.15}$$

$$K_{r+1} H_m + K_m H_r = x^2 R_{r+m}, \tag{3.16}$$

$$G_r H_m - G_m H_r = 2R_{m-r-1}, \tag{3.17}$$

$$G_r R_m - G_{m+1} R_{r-1} = R_{m-r}, \tag{3.18}$$

$$G_r R_m - G_{r+1} R_{m-1} = G_{r-m}, \tag{3.19}$$

$$R_r G_m - R_m G_r = R_{r-m-1}, \tag{3.20}$$

and

$$R_r K_{m+1} - R_m K_{r+1} = 2R_{r-m-1}. \tag{3.21}$$

Many further identities of this type could be given, but these are the ones used in the remainder of the paper.

4. GAMES OF TYPE (i)

Suppose that A is a matrix of type (i) with parameters a and d. Then A is $2n+1 \times 2n+1$, where $n = a+d+1$. To show that there is a matrix D such that $AD = A^*$, as discussed in Section 2, is equivalent to showing that each column of A^* is in the column space of A. However, with the exception of the two irregular columns, every column of A is itself a column of A^*, so we have only to show that columns $a+1$ and $n+a+2$ of A^* are in the column space of A. Let $D_{\cdot j}$ and $A^*_{\cdot j}$ denote the j^{th} column of D and A^*, respectively. What we shall actually exhibit are columns $D_{\cdot j}$ such that $AD_{\cdot j} = \Delta A^*_{\cdot j}$ for $j = a+1$ and $n+a+2$, where

$$\Delta = x(x+2)R_{n-1} \quad (n = a+d+1). \tag{4.0.1}$$

This suffices, in view of the fact that, by Theorem 3.1, $\Delta > 0$ for $x > 2$.

The column $D_{\cdot a+1}$ is defined as follows:

$$\begin{aligned}
d_{i,a+1} &= -2(x+2)T_{a-i} + x(x+2)T_{n-a+i-2} + 2 && \text{for } 1 \le i \le a; \\
d_{a+1,a+1} &= x(x+2)E_{n-1} + 1; \\
d_{i,a+1} &= -2x(x+2)E_{n+a-i} + x^2(x+2)E_{i-a-3} + x && \text{for } a+2 \le i \le n+a+1; \\
d_{n+a+2,a+1} &= -(x+2)T_{n-1} + 1; \\
d_{i,a+1} &= -2(x+2)T_{2n+a+1-i} + x(x+2)T_{i-n-a-3} + 2 && \text{for } n+a+3 \le i \le 2n+1.
\end{aligned} \tag{4.0.2}$$

Theorem 4.1: Let A be a matrix of type (i) as described in Section 2, with parameters a and d. With $D_{\cdot a+1}$, Δ, and A^* as defined above, we have

$$AD_{\cdot a+1} = \Delta A^*_{\cdot a+1}. \tag{4.1.1}$$

Proof: The column $A^*_{\cdot a+1}$ has 1s in rows a, $a+1$, and $2n+1$, $-x$ in row $n+a+1$, and all other elements are 0. Thus, we need to show that the following equations are satisfied:

$$d_{i,a+1}+d_{i+1,a+1}-xd_{n+i+1,a+1}=0 \quad \text{for } 1\le i\le a-1; \tag{4.1.2}$$
$$d_{a,a+1}+2d_{a+1,a+1}-xd_{n+a+1,a+1}=\Delta; \tag{4.1.3}$$
$$d_{a+2,a+1}-xd_{n+a+2,a+1}=\Delta; \tag{4.1.4}$$
$$d_{i,a+1}+d_{i+1,a+1}-xd_{n+i+1,a+1}=0 \quad \text{for } a+2\le i\le n; \tag{4.1.5}$$
$$-xd_{i,a+1}+d_{n+i,a+1}+d_{n+i+1,a+1}=0 \quad \text{for } 1\le i\le a; \tag{4.1.6}$$
$$-xd_{a+1,a+1}+d_{n+a+1,a+1}=-x\Delta; \tag{4.1.7}$$
$$-xd_{a+2,a+1}+2d_{n+a+2,a+1}+d_{n+a+3,a+1}=0; \tag{4.1.8}$$
$$-xd_{i,a+1}+d_{n+i,a+1}+d_{n+i+1,a+1}=0 \quad \text{for } a+3\le i\le n; \tag{4.1.9}$$

$$\sum_{i=1}^{2n+1} d_{i,a+1}=\Delta. \tag{4.1.10}$$

Since the second subscript is $a+1$ in every case, there should be no confusion if we drop it; i.e., we will write d_i for $d_{i,a+1}$. To establish (4.1.2) note that, for $1\le i\le a-1$, we have

$$\begin{aligned} d_1+d_{i+1}-xd_{n+i+1} &= -2(x+2)T_{a-i}+x(x+2)T_{n-a-2+i}+2-2(x+2)T_{a-i-1} \\ &\quad +x(x+2)T_{n-a-1+i}+2+2x^2(x+2)E_{a-i-1}-x^3(x+2)E_{n-a-2+i}-x^2 \\ &= -2(x+2)(T_{a-i}+T_{a-i-1}-x^2E_{a-i-1}) \\ &\quad +x(x+2)(T_{n-a-2+i}+T_{n-a-1+i}-x^2E_{n-a-2+i})+4-x^2 \\ &= (x-2)(x+2)+4-x^2=0, \quad \text{by (3.2).} \end{aligned}$$

For (4.1.3), we have

$$\begin{aligned} d_a+2d_{a+1}-xd_{n+a+1} &= -2(x+2)T_0+x(x+2)T_{n-2}+2+2x(x+2)E_{n-1}+2 \\ &\quad +2x^2(x+2)E_{-1}-x^3(x+2)E_{n-2}-x^2 \\ &= x(x+2)(T_{n-2}+2E_{n-1}-x^2E_{n-2}-1), \quad \text{by (3.0.10) and (3.0.13)} \\ &= \Delta, \quad \text{by (3.2), (3.0.5), and (3.0.3).} \end{aligned}$$

For (4.1.4), note that

$$d_{a+2}-xd_{n+a+2}=x(x+2)(T_{n-1}-2E_{n-2})=\Delta, \quad \text{by (3.0.10), (3.0.5), and (3.0.3).}$$

Both (4.1.5) and (4.1.6) are immediate from (3.0.5).

For (4.1.7), we have

$$-xd_{a+1}+d_{n+a+1}=-x^2(x+2)(E_{n-1}-E_{n-2})=-x\Delta, \quad \text{by (3.0.3) and (3.0.10).}$$

For (4.1.8),

$$-xd_{a+2}+2d_{n+a+2}+d_{n+a+3}=2(x+2)(x^2E_{n-2}-T_{n-1}-T_{n-2}+1)=0, \quad \text{by (3.2).}$$

For (4.1.9), we have, for $a+3\le i\le n$, that

$$\begin{aligned} -xd_i+d_{n+i}+d_{n+i+1} &= 2(x+2)(x^2E_{n+a-i}-T_{n+a+1-i}-T_{n+a-i}) \\ &\quad +x(x+2)(T_{i-a-3}+T_{i-a-2}-x^2E_{i-a-3})+4-x^2 \\ &= 0, \quad \text{by (3.2).} \end{aligned}$$

Finally, for (4.1.10), we have

$$\sum_{i=1}^{2n+1} d_i = -2(x+2)\sum_{i=0}^{a-1} T_i + x(x+2)\sum_{i=n-a-1}^{n-2} T_i + 2a + x(x+2)E_{n-1} + 1$$
$$- 2x(x+2)\sum_{i=-1}^{n-2} E_i + x^2(x+2)\sum_{i=-1}^{n-2} E_i + nx - (x+2)T_{n-1} + 1$$
$$- 2(x+2)\sum_{i=a}^{n-2} T_i + x(x+2)\sum_{i=0}^{n-a-2} T_i + 2(n-a-1)$$
$$= (x^2-4)\sum_{i=0}^{n-2} T_i + x(x^2-4)\sum_{i=0}^{n-2} E_i + x(x+2)E_{n-1} - (x+2)T_{n-1} + n(x+2).$$

With the use of (3.8) and (3.9) we obtain, upon simplification,

$$\sum_{i=1}^{2n+1} d_i = (1 - T_{n-1} - T_{n-2}) + x(E_{n-1} - E_{n-2} - T_{n-1}) + (x^2+2x)E_{n-1}.$$

Then, using (3.2), (3.0.5), and (3.0.3), we have

$$\sum_{i=1}^{2n+1} d_i = -x^2E_{n-2} - 2xE_{n-2} + (x^2+2x)E_{n-1} = (x^2+2x)(E_{n-1} - E_{n-2}) = \Delta,$$

and the proof is complete.

The column $D_{\cdot n+a+2}$ is defined as follows:

$$\begin{aligned} d_{i,\,n+a+2} &= -2(x+2)T_{n-a+i-2} + x(x+2)T_{a-i} + 2 && \text{for } 1 \le i \le a; \\ d_{a+1,\,n+a+2} &= -(x+2)T_{n-1} + 1; \\ d_{i,\,n+a+2} &= -2x(x+2)E_{i-a-3} + x^2(x+2)E_{n+a-i} + x && \text{for } a+2 \le i \le n+a+1; \\ d_{n+a+2,\,n+a+2} &= x(x+2)E_{n-1} + 1; \\ d_{i,\,n+a+2} &= -2(x+2)T_{i-n-a-3} + x(x+2)T_{2n+a+1-i} + 2 && \text{for } n+a+3 \le i \le 2n+1. \end{aligned} \tag{4.1.11}$$

Theorem 4.2: With A, A^*, and Δ as in Theorem 4.1, and $D_{\cdot n+a+2}$ as defined in (4.1.11), we have

$$AD_{\cdot n+a+2} = \Delta A^*_{\cdot n+a+2}. \tag{4.2.1}$$

Proof: The column $A^*_{\cdot n+a+2}$ has $-x$ in row $a+1$, 1 in rows $n+a+1, n+a+2$, and $2n+1$, and 0 in each of the remaining rows. We need to show that the following equations are satisfied:

$$d_{i,\,n+a+2} + d_{i+1,\,n+a+2} - xd_{n+i+1,\,n+a+2} = 0 \quad \text{for } 1 \le i \le a-1; \tag{4.2.2}$$
$$d_{a,\,n+a+2} + 2d_{a+1,\,n+a+2} - xd_{n+a+1,\,n+a+2} = 0; \tag{4.2.3}$$
$$d_{a+2,\,n+a+2} - xd_{n+a+2,\,n+a+2} = -x\Delta; \tag{4.2.4}$$
$$d_{i,\,n+a+2} + d_{i+1,\,n+a+2} - xd_{n+i+1,\,n+a+2} = 0 \quad \text{for } a+2 \le i \le n; \tag{4.2.5}$$
$$-xd_{i,\,n+a+2} + d_{n+i,\,n+a+2} + d_{n+i+1,\,n+a+2} = 0 \quad \text{for } 1 \le i \le a; \tag{4.2.6}$$
$$-xd_{a+1,\,n+a+2} + d_{n+a+1,\,n+a+2} = \Delta; \tag{4.2.7}$$
$$-xd_{a+2,\,n+a+2} + 2d_{n+a+2,\,n+a+2} + d_{n+a+3,\,n+a+2} = \Delta; \tag{4.2.8}$$
$$-xd_{i,\,n+a+2} + d_{n+i,\,n+a+2} + d_{n+i+1,\,n+a+2} = 0 \quad \text{for } a+3 \le i \le n; \tag{4.2.9}$$
$$\sum_{i=1}^{2n+1} d_{i,\,n+a+2} = \Delta. \tag{4.2.10}$$

Again we drop the second subscript, which is $n + a + 2$ in every instance. Thus, we write d_i for $d_{i,n+a+2}$. To show (4.2.2) we note that, for $1 \le i \le a-1$.

$$\begin{aligned} d_i + d_{i+1} - xd_{n+i+1} &= -2(x+2)(T_{n+i-a-2} + T_{n+i-a-1} - x^2 E_{n+i-a-2}) \\ &\quad + x(x+2)(T_{a-i} + T_{a-i-1} - x^2 E_{a-i-1}) + 4 - x^2 \\ &= 0, \quad \text{by (3.2)}. \end{aligned}$$

For (4.2.3),

$$\begin{aligned} d_a + 2d_{a+1} - xd_{n+a+1} &= -2(x+2)(T_{n-2} + T_{n-1} - x^2 E_{n-2}) + x(x+2) + 4 - x^2 \\ &= 0, \quad \text{by (3.2)}. \end{aligned}$$

For (4.2.4), $d_{a+2} - xd_{n+a+2} = -x^2(x+2)(E_{n-1} - E_{n-2}) = -x\Delta$, by (3.0.3).

For (4.2.5), note that, for $a+2 \le i \le n$,

$$\begin{aligned} d_i + d_{i+1} - xd_{n+i+1} &= -2x(x+2)(E_{i-a-3} + E_{i-a-2} - T_{i-a-2}) + x^2(x+2)(E_{n+a-i} + E_{n+a-i-1} - T_{n+a-i}) \\ &= 0, \quad \text{by (3.0.5)}. \end{aligned}$$

For (4.2.6), we have, for $1 \le i \le a$, that

$$\begin{aligned} -xd_i + d_{n+i} + d_{n+i+1} &= 2x(x+2)(T_{n+i-a-2} - E_{n+i-a-3} - E_{n+i-a-2}) + x^2(x+2)(E_{a-i} + E_{a-i-1} - T_{a-i}) \\ &= 0, \quad \text{by (3.0.5)}. \end{aligned}$$

For (4.2.7), $-xd_{a+1} + d_{n+a+1} = x(x+2)(T_{n-1} - 2E_{n-2}) = \Delta$, by (3.0.5) and (3.0.3).

For (4.2.8) we have, using (3.0.10) and (3.0.13),

$$\begin{aligned} -xd_{a+2} + 2d_{n+a+2} + d_{n+a+3} &= -x^3(x+2)E_{n-2} + 2x(x+2)E_{n-1} + x(x+2)T_{n-2} + 4 - x^2 - 2(x+2) \\ &= x(x+2)(-x^2 E_{n-2} + 2E_{n-1} + T_{n-2} - 1) \\ &= \Delta, \quad \text{by; (3.2), (3.0.5), and (3.0.3)}. \end{aligned}$$

For (4.2.9), note that, for $a+3 \le i \le n$,

$$\begin{aligned} -xd_i + d_{n+i} + d_{n+i+1} &= 2(x+2)(x^2 E_{i-a-3} - T_{i-a-3} - T_{i-a-2}) \\ &\quad + x(x+2)(-x^2 E_{n+a-i} + T_{n+a+1-i} + T_{n+a-i}) + 4 - x^2 \\ &= 0, \quad \text{by (3.2)}. \end{aligned}$$

Finally, (4.2.10) follows from (4.1.10) since the elements of $D_{\cdot n+a+2}$ are precisely those of $D_{\cdot a+1}$ but reordered. This completes the proof.

We turn now to the solution of (2.4). Let U be the column with components

$$\begin{aligned} u_i &= G_d K_{a+1-i} && \text{for } 1 \le i \le a; \\ u_{a+1} &= G_d; && \\ u_i &= xG_a G_{i-a-2} && \text{for } a+2 \le i \le n+1; \\ u_i &= xG_{n+a+1-i} G_d && \text{for } n+1 \le i \le n+a+1; \\ u_{n+a+2} &= G_a; && \\ u_i &= K_{i-n-a-2} G_a && \text{for } n+a+3 \le i \le 2n+1. \end{aligned} \tag{4.2.11}$$

(Note that u_{n+1} occurs twice but that the two expressions agree.)

Theorem 4.3: With U as defined by (4.2.11), the column $P^t = U/(x+2)R_{n-1}$ satisfies (2.4), namely $AP^t = (0, 0, \ldots, 0, 1)^t$, for the matrix A of type (i), and has all components positive for $x > 2$. The vector P is thus the unique optimal strategy for the row player in the reduced game of type (i).

Proof: That all components are positive for $x > 2$ is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $AU = (0, 0, \ldots, 0, \Delta)^t$, where $\Delta = (x+2)R_{n-1}$.

For $1 \le i \le a-1$, we have

$$A_{i\cdot}U = u_i + u_{i+1} - xu_{n+i+1} = G_d(K_{a+1-i} + K_{a-i} - x^2 G_{a-i}) = 0, \quad \text{by (3.3).}$$

Also,

$$A_{a\cdot}U = u_a + 2u_{a+1} - xu_{n+a+1} = G_d(K_1 + 2 - x^2) = 0, \quad \text{by (3.0.15),}$$

and

$$A_{a+1\cdot}U = u_{a+2} - xu_{n+a+2} = xG_a G_0 - xG_a = 0,$$

since $G_0 = 1$.

For $a+2 \le i \le n$,

$$A_{i\cdot}U = u_i + u_{i+1} - xu_{n+i+1} = xG_a(G_{i-a-2} + G_{i-a-1} - K_{i-a-1}),$$

and, for $n+1 \le i \le n+a$,

$$A_{i\cdot}U = -xu_{i-n} + u_i + u_{i+1} = xG_d(-K_{n+a+1-i} + G_{n+a+1-i} + G_{n+a-i}),$$

and both of these are 0 by (3.0.7).

Next,

$$A_{n+a+1\cdot}U = -xu_{a+1} + u_{n+a+1} = -xG_d + xG_d = 0,$$

and

$$A_{n+a+2\cdot}U = -xu_{a+2} + 2u_{n+a+2} + u_{n+a+3} = G_a(-x^2 + 2 + K_1) = 0, \quad \text{by (3.0.15).}$$

For $n+a+3 \le i \le 2n$, we have, by (3.3),

$$A_{i\cdot}U = -xu_{i-n} + u_i + u_{i+1} = G_a(-x^2 G_{i-n-a-2} + K_{i-n-a-2} + K_{i-n-a-1}) = 0.$$

Finally, using (3.0.7), (3.0.4), (3.0.14), and (3.0.12), we have

$$\begin{aligned} A_{2n+1\cdot}U = \sum_{i=1}^{2n+1} u_i &= G_d\left(1 + \sum_{i=1}^{a} K_i\right) + xG_a \sum_{i=0}^{d} G_i + xG_d \sum_{i=0}^{a-1} G_i + G_a\left(1 + \sum_{i=1}^{d} K_i\right) \\ &= (G_d H_a + G_a H_d) + x(G_a R_d + G_d R_{a-1}) \end{aligned}$$

(recall that $d = n-a-1$), and in view of (3.10.1) and (3.11.1) this is equal to $(x+2)R_{n-1}$, as claimed. This completes the proof.

For the column player's optimal strategy, we use the vector $W = (w_1, w_2, \ldots, w_{2n+1})$ defined by (4.3.1) below:

$$\begin{aligned}
w_i &= x(x^2-4)R_{a-i}H_d && \text{for } 1 \le i \le a;\\
W_{a+1} &= 2H_a + xH_d;\\
w_i &= (x^2-4)H_aH_{i-a-2} && \text{for } a+2 \le i \le n+a+1;\\
w_i &= (x^2-4)H_{n+a+1-i}H_d && \text{for } n+1 \le i \le n+a+1;\\
w_{n+a+2} &= xH_a + 2H_d;\\
w_i &= x(x^2-4)H_aR_{i-n-a-3} && \text{for } n+a+3 \le i \le 2n+1.
\end{aligned} \tag{4.3.1}$$

Theorem 4.4: For $x > 2$, the vector $Q = W / x(x+2)R_{n-1}$, where W is defined by (4.3.1), has all components positive and satisfies (2.1b) for the batrix B of type (i). This is therefore the unique optimal strategy for the column player.

Proof: The proof is very similar to that of the preceding theorem, and we omit the details.

The game value, $V_{(\mathrm{i})}$, for the reduced game of type (i) is now easily computed as well. It is given by the product $PB_{\cdot j}$ for any column $B_{\cdot j}$ of the payoff matrix. Using the middle column, we have

$$V_{(\mathrm{i})} = PB_{\cdot n+1} = \left(-\sum_{i=1}^{n} u_i + \sum_{i=n+2}^{2n+1} u_i\right)\Big/(x+2)R_{n-1},$$

and with the use of (3.0.7), (3.0.4), (3.17), and (3.18), we obtain (4.5.1) below.

Theorem 4.5: For $x > 2$, the game value $V_{(\mathrm{i})}$ for the reduced game of type (i) is given by

$$V_{(\mathrm{i})} = \frac{(x-2)R_{a-d-1}}{(x+2)R_{a+d}}. \tag{4.5.1}$$

Moreover,

$$V_{(\mathrm{i})} > 0,\ V_{(\mathrm{i})} = 0,\ \text{or } V_{(\mathrm{i})} < 0 \ \text{ according as } \ a > d,\ a = d,\ \text{or } a < d. \tag{4.5.2}$$

Proof: The assertion (4.5.2) follows from Theorem 3.1 and (3.0.11).

5. GAMES OF TYPE (ii)

In a matrix A of type (ii), only columns $c+2, n+c+2$, and $n+c+3$ differ from the corresponding columns of A^*, so to show nonsingularity of A it would suffice to show that these three columns of A^* lie in the column space of A. However, we can simplify the problem further by the observation that the type (ii) matrix A with parameters c, d differs from the type (i) matrix A' with parameters $a' = c+1, d' = d$ only in column $n+c+2 = n+a'+1$, and in this column, A' agrees with A^*. Thus, it suffices to show that $A^*_{\cdot n+c+2}$ lies in the column space of the type (ii) matrix A. To that end, we use the column D defined by (5.0.1) below, and show that $AD = xG_{n-1}A^*_{\cdot n+c+2}$, which suffices in view of Theorem 3.1.

$$\begin{aligned}
d_i &= -K_{n+i-c-2} && \text{for } 1 \le i \le c+1;\\
d_{c+2} &= H_{n-1};\\
d_i &= -xG_{i-c-3} && \text{for } c+3 \le i \le n+c+1;\\
d_{n+c+2} &= xR_{n-1};\\
d_{n+c+3} &= -1;\\
d_i &= -K_{i-n-c-3} && \text{for } n+c+4 \le i \le 2n+1.
\end{aligned} \tag{5.0.1}$$

Theorem 5.1: Let A be a matrix of type (ii) with parameters c and d, and let A^* be the associated matrix of the same dimensions as A as described in Section 2. With D as defined in (5.0.1), we have

$$AD = xG_{n-1}A^*_{\cdot n+c+2}. \tag{5.1.1}$$

Proof: The column $A^*_{\cdot n+a+2}$ has $-x$ in row $c+1$, 1 in rows $n+c+1, n+c+2$, and $2n+1$, and 0 in each of the remaining rows. We need only show, therefore, that the following conditions are fulfilled:

$$d_i + d_{i+1} - xd_{n+i+1} = 0 \quad \text{for } 1 \le i \le c; \tag{5.1.2}$$
$$d_{c+1} + 2d_{c+2} - xd_{n+c+2} = -x^2 G_{n-1}; \tag{5.1.3}$$
$$d_{c+3} - xd_{n+c+3} = 0; \tag{5.1.4}$$
$$d_i + d_{i+1} - xd_{n+i+1} = 0 \quad \text{for } c+3 \le i \le n; \tag{5.1.5}$$
$$-xd_{i-n} + d_i + d_{i+1} = 0 \quad \text{for } n+1 \le i \le n+c; \tag{5.1.6}$$
$$-xd_{c+1} + d_{n+c+1} = xG_{n-1}; \tag{5.1.7}$$
$$-xd_{c+2} + 2d_{n+c+2} = xG_{n-1}; \tag{5.1.8}$$
$$-xd_{c+3} + 2d_{n+c+3} + d_{n+c+4} = 0; \tag{5.1.9}$$
$$-xd_{i-n} + d_i + d_{i+1} = 0 \quad \text{for } n+c+4 \le i \le 2n; \tag{5.1.10}$$
$$\sum_{i=1}^{2n+1} d_i = xG_{n-1}. \tag{5.1.11}$$

For (5.1.2) we have, for $1 \le i \le c$,

$$d_i + d_{i+1} - xd_{n+i+1} = -K_{n+i-c-2} - K_{n+i-c-1} + x^2 G_{n+i-c-2} = 0, \quad \text{by (3.3).}$$

For (5.1.3),

$$\begin{aligned} d_{c+1} - 2d_{c+2} - xd_{n+c+2} &= -K_{n-1} + 2H_{n-1} - x^2 R_{n-1} \\ &= x^2 G_{n-1}, \quad \text{by (3.3), (3.4), and (3.0.7).} \end{aligned}$$

For (5.1.4), $d_{c+3} - xd_{n+c+3} = -xG_0 + x = 0$, by (3.0.12).

For (5.1.5), note that, for $c+3 \le i \le n$,

$$d_i + d_{i+1} - xd_{n+i+1} = -x(G_{i-c-3} + G_{i-c-2} - K_{i-c-2}) = 0, \quad \text{by (3.0.7).}$$

For (5.1.6), we have, for $n+1 \le i \le n+c$,

$$-xd_{i-n} + d_i + d_{i+1} = x(K_{i-c-2} - G_{i-c-3} - G_{i-c-2}) = 0, \quad \text{by (3.0.7).}$$

For (5.1.7), $-xd_{c+1} + d_{n+c+1} = x(K_{n-1} - G_{n-2}) = xG_{n-1}$, by (3.0.7).

For (5.1.8), observe that

$$-xd_{c+2} + 2d_{n+c+2} = x(-H_{n-1} + 2R_{n-1}) = xG_{n-1}, \quad \text{by (3.0.6) and (3.0.4).}$$

For (5.1.9), we have

$$-xd_{c+3} + 2d_{n+c+3} + d_{n+c+4} = x^2 G_0 - 2 - K_1 = 0, \quad \text{by (3.0.12) and (3.0.15).}$$

For (5.1.10), we note that, for $n+c+4 \le i \le 2n$,

$$-xd_{i-n} + d_i + d_{i+1} = x^2 G_{i-n-c-3} - K_{i-n-c-3} - K_{i-n-c-2} = 0, \quad \text{by (3.3).}$$

Finally, for (5.1.11), we have

$$\sum_{i=1}^{2n+1} d_i = -\sum_{i=1}^{n-1} K_i - 1 + H_{n-1} - x\sum_{i=0}^{n-2} G_i + xR_{n-1}.$$

From (3.0.7) and (3.0.15), we obtain

$$\sum_{i=1}^{n-1} K_i = H_{n-1} - 1,$$

and from (3.0.4) and (3.0.11),

$$\sum_{i=0}^{n-2} G_i = R_{n-2},$$

so that

$$\sum_{i=1}^{2n+1} d_i = x(R_{n-1} - R_{n-2}) = xG_{n-1},$$

and the proof is complete.

We turn now to the solution of (2.4) for matrices A of type (ii). Let D be the column with components as given in (5.1.12) below.

$$\begin{aligned}
d_i &= (x^2-4)G_d H_{c+1-i} && \text{for } 1 \le i \le c+1; \\
d_{c+2} &= 2G_d; && \\
d_i &= x(x^2-4)R_c G_{i-c-3} && \text{for } c+3 \le i \le n+1; \\
d_i &= x(x^2-4)R_{n+1+c-i} G_d && \text{for } n+1 \le i \le n+c+1; \\
d_{n+c+2} &= xG_d; && \\
d_{n+c+3} &= (x^2-4)R_c; && \\
d_i &= (x^2-4)R_c K_{i-n-c-3} && \text{for } n+c+4 \le i \le 2n+1.
\end{aligned} \tag{5.1.12}$$

Note again that the two expressions for d_{n+1} agree.

Theorem 5.2: Let A be the matrix of type (ii) with parameters c and d. Let $P^t = D/(x+2)G_{n-1}$, where D is as defined by (5.1.12). Then P satisfies (2.4), namely $AP^t = (0, 0, \dots, 0, 1)^t$, and has all components positive for $x > 2$.

Proof: That all components are positive for $x > 2$ is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $AD = (0, 0, \dots, 0, \Delta)$, where $\Delta = (x+2)G_{n-1}$. Let $A_{i\cdot}$ denote the i^{th} row of A.

For $1 \le i \le c$,

$$A_{i\cdot}D = d_i + d_{i+1} - xd_{n+i+1} = (x^2-4)G_d(H_{c+1-i} + H_{c-i} - x^2 R_{c-i}) = 0, \quad \text{by (3.4).}$$

Also,

$$A_{c+1\cdot}D = d_{c+1} + 2d_{c+2} - xd_{n+c+2} = (x^2-4)G_d - x^2 G_d = 0.$$

Next,

$$A_{c+2}.D = d_{c+3} - xd_{n+c+3} = x(x^2-4)R_cG_0 - x(x^2-4)R_c = 0.$$

For $c+3 \le i \le n$,

$$A_i.D = d_i + d_{i+1} - xd_{n+i+1} = x(x^2-4)R_c(G_{i-c-3} + G_{i-c-2} - K_{i-c-2}) = 0, \quad \text{by (3.0.7).}$$

For $n+1 \le i \le n+c$,

$$A_i.D = -xd_{i-n} + d_i + d_{i+1} = x(x^2-4)(-H_{n+c+1-i} + R_{n+c+1-i} + R_{n+c-i}) = 0, \quad \text{by (3.0.6).}$$

We have

$$A_{n+c+1}.D = -xd_{c+1} + d_{n+c+1} = x(x^2-4)G_d(-H_0 + R_0) = 0,$$
$$A_{n+c+2}.D = -xd_{c+2} + 2d_{n+c+2} = 0,$$

and

$$A_{n+c+3}.D = -xd_{c+3} + 2d_{n+c+3} + d_{n+c+4} = (x^2-4)R_c(-x^2G_0 + 2 + K_1)$$
$$= 0, \quad \text{by (3.0.12) and (3.0.15).}$$

For $n+c+4 \le i \le 2n$,

$$A_i.D = -xd_{i-n} + d_i + d_{i+1} = (x^2-4)R_c(-x^2G_{i-n-c-3} + K_{i-n-c-3} + K_{i-n-c-2}) = 0, \quad \text{by (3.3).}$$

Finally,

$$A_{2n+1}.D = \sum_{i=1}^{2n+1} d_i$$
$$= (x^2-4)G_d\sum_{i=0}^{c} H_i + 2G_d + x(x^2-4)R_c\sum_{i=0}^{d} G_i + x(x^2-4)G_d\sum_{i=0}^{c-1} R_i$$
$$+ xG_d + (x^2-4)R_c\left(1 + \sum_{i=1}^{d} K_i\right)$$
$$= (x^2-4)G_dT_c + (x+2)G_d + x(x^2-4)R_cR_d + x(x^2-4)G_dE_{c-1} + (x^2-4)R_cH_d,$$

using, in turn, (3.0.6), (3.0.13), (3.0.11), (3.0.3), (3.0.10), (3.0.7), and (3.0.14). Upon factoring out $(x+2)$ and separating into even and odd parts, we obtain

$$\frac{1}{x+2}\sum_{i=1}^{2n+1} d_i = (-2(G_dT_c + H_dR_c) + G_d + x^2(R_cR_d + E_{c-1}G_d))$$
$$+ x((G_dT_c + H_dR_c) - 2(R_cR_d + E_{c-1}G_d)).$$

The odd part is 0, since $G_d(T_c - 2E_{c-1}) + R_c(H_d - 2R_d) = G_dR_c - R_cG_d$, by (3.0.5), (3.0.3), (3.0.6), and (3.0.4). Thus, we have

$$\frac{1}{x+2}\sum_{i=1}^{2n+1} d_i = (x^2-4)(R_cR_d + E_{c-1}G_d) + G_d$$
$$= (G_{c+1} - G_c)R_d + G_cG_d, \quad \text{by (3.5) and (3.6),}$$
$$= G_{c+1}R_d - G_cR_{d-1}$$
$$= G_{c+d+1} = G_{n-1}, \quad \text{by (3.12).}$$

This completes the proof.

For the optimal strategy for the column player, we use the vector W with components as given in (5.2.1) below.

$$\begin{aligned} w_i &= x(x^2-4)G_{c+1-i}H_d && \text{for } 1 \le i \le c+1; \\ w_{c+2} &= 2K_{c+1}; \\ w_i &= (x^2-4)K_{c+1}H_{i-c-3} && \text{for } c+3 \le i \le n+1; \\ w_i &= (x^2-4)K_{n+c+2-i}H_d && \text{for } n+1 \le i \le n+c+1; \\ w_{n+c+2} &= (x^2-4)H_d; \\ w_{n+c+3} &= xK_{c+1}; \\ w_i &= x(x^2-4)K_{c+1}R_{i-n-c-4} && \text{for } n+c+4 \le i \le 2n+1. \end{aligned} \tag{5.2.1}$$

Theorem 5.3: For $x > 2$, the vector $Q = W / x(x+2)G_{n-1}$, where W is the vector defined by (5.2.1), has all components positive, and satisfies (2.1b) for the matrix B of type (ii). Therefore, this is the unique optimal strategy for the column player in the game with payoff matrix B.

The proof is similar to the proof of the preceding theorem, and we omit the details.

The middle column, $B_{\cdot n+1}$ is the same for all four types of reduced matrix, and we use it again to compute the game value, $V_{\text{(ii)}}$. With D as given by (5.1.12), we have

$$V_{\text{(ii)}} = \left(-\sum_{i=1}^{n} d_i + \sum_{i=n+2}^{2n+1} d_i\right) \Bigg/ (x+2)G_{n-1},$$

and with the use of (3.0.6), (3.0.4), (3.0.3), (3.0.7), (3.5), (3.6), (3.7), (3.17), and (3.19), we obtain (5.4.1) below.

Theorem 5.4: For $x > 2$, the game value, $V_{\text{(ii)}}$, for the reduced game of type (ii) is given by

$$V_{\text{(ii)}} = \frac{(x-2)G_{c-d}}{(x+2)G_{c+d+1}}, \tag{5.4.1}$$

and

$$V_{\text{(ii)}} > 0 \text{ for all } c \text{ and } d. \tag{5.4.2}$$

Proof: The assertion (5.4.2) follows from (3.0.12) and Theorem 3.1.

6. GAMES OF TYPE (iii)

The payoff matrix for a game of type (iii) is sufficiently closely related to that for a game of type (ii) that we may use our results from Section 5 to obtain the corresponding theorems here. The key observation is the following.

Remark 6.1: Let B be the payoff matrix for a game of type (iii) with parameters a and b, and let B' be the payoff matrix for a game of type (ii) with parameters $c' = b$ and $d' = a$. If we change all signs in B, transpose about the main diagonal, and then transpose about the lower left to upper right diagonal, we obtain the matrix B'.

The matrix $-B^t$ obtained after the first two steps in Remark 6.1 is the payoff matrix of the game B with the roles of the players reversed. The third step obviously also preserves rank, so uniqueness of solutions P, Q, and V to

$$PB = (V, V, \ldots, V) \tag{6.1.1}$$

and

$$BQ^t = (V, V, \ldots, V)^t \tag{6.1.2}$$

follow from uniqueness of solutions to

$$P'B' = (V', V', \ldots, V') \tag{6.1.3}$$

and

$$B'Q'^t = (V', V', \ldots, V'). \tag{6.1.4}$$

Moreover, the transposition of $-B^t$ about its counterdiagonal sends row i of $-B^t$ to column $2n+1-i$ of B', and column j of $-B^t$ to row $2n+1-j$ of B'. Thus we see that, if P', Q', and V' satisfy (6.1.3) and (6.1.4), and we define P to be the vector Q' with the order of the elements reversed, Q to be P' reversed, and $V = -V'$, then P, Q, and V satisfy (6.1.1) and (6.1.2). We summarize this in the next theorem.

Theorem 6.2: Let B be the payoff matrix of a game of type (iii), with $x > 2$ and B' the associated payoff matrix of type (ii) as described above. Let P', Q', and V' be, respectively, the optimal strategy for the row player, the optimal strategy for the column player, and the game value for B', and let P and Q be, respectively, Q' reversed and P' reversed. Then P and Q are the optimal strategies for the row and column players, respectively, for the game B, and the game value, $V_{(\text{iii})}$, is given by

$$V_{(\text{iii})} = -V' = -\frac{(x-2)G_{b-a}}{(x+2)G_{b+a+1}}. \tag{6.2.1}$$

The game value is negative for all values of b and a.

7. GAMES OF TYPE (iv)

The type (iv) matrix A, with parameters c and b, is a $2n+1 \times 2n+1$ matrix, where $n = c + b + 2$. In this matrix A, only column $c + 1$ differs from the corresponding column of A', where A' is the type (iii) matrix with parameters $a' = c$ and $b' = b$. We shall establish nonsingularity of A by exhibiting a column D such that

$$AD = A'_{\cdot c+1}\Delta, \tag{7.0.1}$$

where $A'_{\cdot c+1}$ is column $c + 1$ of A' and $\Delta = x(x+2)R_{n-1}$. The column D is defined by (7.0.2) below.

$$\begin{aligned}
d_i &= -x(x+2)H_{b+i} && \text{for } 1 \le i \le c; \\
d_{c+1} &= x(x+2)G_{n-1}; && \\
d_{c+2} &= 2x(x+2)E_{n-2} - x; && \\
d_i &= -x^2(x+2)R_{i-c-3} && \text{for } c+3 \le i \le n+c+1; \\
d_{n+c+2} &= x^2(x+2)E_{n-2} + x; && \\
d_i &= -x(x+2)H_{i-n-c-3} && \text{for } n+c+3 \le i \le 2n+1.
\end{aligned} \tag{7.0.2}$$

Theorem 7.1: Let A be a matrix of type (iv) with parameters c and b, and $x > 2$. Let A' be the matrix of type (iii) with parameters $a' = c$ and $b' = b$. Then the column D defined by (7.0.2) satisfies (7.0.1) and thus A is nonsingular.

Proof: The column $A'_{\cdot c+1}$ has a 2 in row c (if $c > 0$), $-x$ in row $n+c+1$, 1 in the last row, and 0 in all other rows.

For $1 \le i \le c$, $A_{i\cdot}D = d_i + d_{i+1} - xd_{n+i+1}$. If $i < c$, this is $x(x+2)(-H_{b+i} - H_{b+i+1} + x^2 R_{b+i})$, which is 0 by (3.4). If $i = c$, we have $A_i D = x(x+2)(-H_{n-2} + G_{n-1} - x^2 R_{n-2}) = 2x(x+2)R_{n-1} = 2\Delta$, by (3.4), (3.0.4), and (3.0.6).

For rows $c+1$ and $c+2$, we have

$$A_{c+1\cdot}D = d_{c+1} + 2d_{c+2} - xd_{n+c+2} = x(x+2)(G_{n-1} + 4E_{n-2} - x^2 E_{n-2} - 1) = 0, \quad \text{by (3.6)},$$

and

$$A_{c+2\cdot}D = d_{c+3} - xd_{n+c+3} = x^2(x+2)(-R_0 + H_0) = 0.$$

For $c+4 \le i \le n$,

$$A_{i\cdot}D = d_i + d_{i+1} - xd_{n+i+1} = x^2(x+2)(-R_{i-c-3} - R_{i-c-2} + H_{i-c-2}) = 0, \quad \text{by (3.0.6)}.$$

For $n+1 \le i \le n+c$,

$$A_{i\cdot}D = -xd_{i-n} + d_i + d_{i+1} = x^2(x+2)(H_{i+b-n} - R_{i-c-3} - R_{i-c-2}) = 0, \quad \text{by (3.0.6)},$$

since $n = b+c+2$.

With $i = n+c+1$, we have

$$A_{n+c+1\cdot}D = -xd_{c+1} + d_{n+c+1} = -x^2(x+2)(G_{n-1} + R_{n-2}) = -x\Delta, \quad \text{by (3.0.4)},$$

and

$$A_{n+c+2\cdot}D = -xd_{c+2} + 2d_{n+c+2} + d_{n+c+3} = x^2 + 2x - x(x+2)H_0 = 0.$$

For $n+c+3 \le i \le 2n$,

$$A_{i\cdot}D = -xd_{i-n} + d_i + d_{i+1} = x(x+2)(-x^2 R_{i-n-c-3} + H_{i-n-c-3} + H_{i-n-c-2}) = 0, \quad \text{by (3.4)}.$$

Finally,

$$A_{2n+1\cdot}D = \sum_{i=1}^{2n=1} d_i = x(x+2)\left(-\sum_{i=0}^{n-2} H_i + G_{n-1} + (x+2)E_{n-2} - x\sum_{i=0}^{n-2} R_i\right).$$

By (3.0.3) and (3.0.10), $\sum_{i=0}^{n-2} R_i = E_{n-2}$, and by (3.0.6) and (3.0.13), $\sum_{i=0}^{n-2} H_i = T_{n-2}$. Thus,

$$\sum_{i=1}^{2n+1} d_i = x(x+2)(-T_{n-2} + G_{n-1} + 2E_{n-2}),$$

and with the help of (3.0.5), (3.0.4), and (3.0.3), this is easily seen to be equal to Δ. This completes the proof.

We turn now to the solution of (2.4) for the matrix A of type (iv). Let D be the column with components as defined in (7.1.1) below.

$$\begin{aligned} d_i &= (x^2-4)H_{c+1-i}R_b && \text{for } 1 \le i \le c+1; \\ d_{c+2} &= xR_c + 2R_b; && \\ d_i &= x(x^2-4)R_cR_{i-c-3} && \text{for } c+3 \le i \le n+1; \\ d_i &= x(x^2-4)R_{n+1+c-i}R_b && \text{for } n+1 \le i \le n+c+1; \\ d_{n+c+2} &= 2R_c + xR_b; && \\ d_i &= (x^2-4)H_{i-n-c-3}R_c && \text{for } n+c+3 \le i \le 2n+1. \end{aligned} \tag{7.1.1}$$

Theorem 7.2: Let A be the matrix of type (iv) with parameters c and b. Let $P^t = D/(x+2)R_{n-1}$, where D is defined by (7.1.1). Then P satisfies (2.4) and has all components positive for $x > 2$. Thus, P is the unique optimal strategy vector for the row player in the game of type (iv).

Proof: That all components are positive is clear from Theorem 3.1. To prove that (2.4) is satisfied, we show that $AD = (0, 0, \ldots, 0, \Delta)$, where $\Delta = (x+2)R_{n-1}$.

For $1 \le i \le c$,

$$A_{i\cdot}D = d_i + d_{i+1} - xd_{n+i+1} = (x^2-4)R_b(H_{c+1-i} + H_{c-i} - x^2R_{c-i}) = 0, \quad \text{by (3.4)}.$$

For rows $c+1$ and $c+2$, we have

$$A_{c+1\cdot}D = d_{c+1} + 2d_{c+2} - xd_{n+c+2} = (x^2-4)H_0R_b + 2xR_c + 4R_b - 2xR_c - x^2R_b = 0,$$

and

$$A_{c+2\cdot}D = d_{c+3} - xd_{n+c+3} = x(x^2-4)R_c(R_0 - H_0) = 0,$$

since $H_0 = R_0 = 1$ by; (3.0.11) and (3.0.14).

For $c+3 \le i \le n$,

$$A_{i\cdot}D = d_i + d_{i+1} - xd_{n+i+1} = x(x^2-4)R_c(R_{i-c-3} + R_{i-c-2} - H_{i-c-2} = 0, \quad \text{by (3.0.6)}.$$

For $n+1 \le i \le n+c$,

$$A_{i\cdot}D = -xd_{i-n} + d_i + d_{i+1} = x(x^2-4)R_b(-H_{n+c+1-i} + R_{n+c+1-i} + R_{n+c-i}) = 0, \quad \text{by (3.0.6)}.$$

For the next two rows, we have

$$A_{n+c+1\cdot}D = xd_{c+1} + d_{n+c+1} = x(x^2-4)R_b(H_0 - R_0) = 0,$$

and

$$A_{n+c+2\cdot}D = -xd_{c+2} + 2d_{n+c+2} + d_{n+c+3} = (-x^2+4)R_c + (-2x+2x)R_b + (x^2-4)H_0R_c = 0.$$

For $n+c+3 \le i \le 2n$,

$$A_{i\cdot}D = -xd_{i-n} + d_i + d_{i+1} = (x^2-4)R_c(-x^2R_{i-n-c-3} + H_{i-n-c-3} + H_{i-n-c-2}) = 0, \quad \text{by (3.4)}.$$

Finally, using (3.0.6), (3.0.13), (3.0.3), and (3.0.10), we have

$$\begin{aligned}A_{2n+1}.D = \sum_{i=1}^{2n+1} d_i &= (x^2-4)R_b\sum_{i=0}^{c} H_i + (x+2)(R_c+R_b)\\ &+ x(x^2-4)R_c\sum_{i=0}^{b} R_i + x(x^2-4)R_b\sum_{i=0}^{c-1} R_i + (x^2-4)R_c\sum_{i=0}^{b} H_i\\ &= (x^2-4)(R_bT_c+R_cT_b)+(x+2)(R_c+R_b)+x(x^2-4)(R_cE_b+R_bE_{c-1}).\end{aligned}$$

Upon factoring out $(x+2)$ and separating into even and odd parts, we obtain

$$\frac{1}{x+2}\sum_{i=1}^{2n+1} d_i = (R_b(x^2E_{c-1}-2T_c+1)+R_c(x^2E_b-2T_b+1))+x(R_b(T_c-2E_{c-1})+R_c(T_b-2E_b)).$$

The odd part is easily seen to be 0 using (3.0.5) and (3.0.3), and with the help of (3.2), (3.0.6), and (3.13), we see that the even part is R_{b+c+1}. Since $n=b+c+2$, we have

$$A_{2n+1}.D = (x+2)R_{n-1},$$

and the proof is complete.

To describe the optimal strategy for the column player, we use the vector W defined in (7.2.1) below.

$$\begin{aligned} w_i &= xG_{c+1-i}K_{b+1} && \text{for } 1\le i\le c+1;\\ w_{c+2} &= K_{c+1}; &&\\ w_i &= K_{c+1}K_{i-c-2} && \text{for } c+3\le i\le n+1;\\ w_i &= K_{n+c+2-i}K_{b+1} && \text{for } n+1\le i\le n+c+1;\\ w_{n+c+2} &= K_{b+1}; &&\\ w_i &= xK_{c+1}G_{i-n-c-3} && \text{for } n+c+3\le i\le 2n+1.\end{aligned}\tag{7.2.1}$$

Theorem 7.3: The vector $Q = W/x(x+2)R_{n-1}$, where W is defined by (7.2.1), has all components positive for $x>2$, and satisfies (2.1*b*) for the matrix B of type (iv). Therefore, this is the unique optimal strategy for the column player in the game with payoff matrix B.

The proof is straightforward and is left to the reader.

With D as given by (7.1.1), we again express the game value $V_{(\text{iv})}$ in the form

$$V_{(\text{iv})} = \left(-\sum_{i=1}^{n} d_i + \sum_{i=n+2}^{2n+1} d_i\right)\Bigg/(x+2)R_{n-1},$$

and using (3.0.6), (3.0.3), (3.7), (3.6), (3.21), and (3.20), we obtain (7.4.1) below.

Theorem 7.4: The game value $V_{(\text{iv})}$ for the reduced game of type (iv) with $x>2$ is given by

$$V_{(\text{iv})} = \frac{(x-2)R_{b-c-1}}{(x+2)R_{b+c+1}}.\tag{7.4.1}$$

Moreover,

$$V_{(\text{iv})}>0,\ V_{(\text{iv})}=0, \text{ or } V_{(\text{iv})}<0 \ \text{ according as } \ b>c, b=c, \text{ or } b<c.\tag{7.4.2}$$

With the theorems of Sections 4-7 we have now established the irreducibility of the Silverman games in the four classes of odd order games which arise in Chapter 8 of [7], and have given game values and optimal strategies explicitly in terms of the various parameters involved.

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