CONGRUENCES FOR A WIDE CLASS OF INTEGERS BY USING GESSEL'S METHOD

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1. INTRODUCTION AND PREPARATORY RESULTS

Let $P_n, n = 0, 1, 2, \ldots$, be a sequence of integers that is defined by its exponential generating function $f(x)$ as

$$\sum_{n=0}^{\infty} P_n x^n / n! = f(x). \tag{1}$$

That is $f(x)$ is a Hurwitz series in x.

As regards Bell numbers $[f(x) = \exp\{\exp\{x\} - 1\}]$, Lunnon, Pleasants, & Stephens [4] and Gessel [1] showed that, for each positive integer n, there exist integers $a_0, a_1, \ldots, a_{n-1}$ such that, for all $m \geq 0, n \geq 0$,

$$P_{m+n} + a_{n-1}P_{m+n-1} + \cdots + a_0 P_m \equiv 0 \pmod{n!}. \tag{2}$$

Also, as regards tangent numbers $[f(x) = \tan x]$, Ira Gessel [1] showed that, for each positive integer n, there exist integers $b_1, b_2, \ldots, b_{n-1}$ such that, for all $m \geq 0, n \geq 1$,

$$P_{m+n} + b_{n-1}P_{m+n-1} + \cdots + b_1 P_{m+1} \equiv 0 \pmod{(n-1)!n!}.$$

In the same paper, congruences similar to the above are obtained concerning the derangement num-bers and the numbers defined by $f(x) = (2 - \exp\{x\})^{-1}$ and $f(x) = \exp\{x + x^2/2\}$. In the same area of research, Kyriakoussis [3] proved the congruence (2) in the case in which

$$f(x) = \exp\{g(x)\}, \text{ for } g(x) = \sum_{j=1}^{\infty} c_j x^j / j,$$

where the $c_j, j = 1, 2, \ldots$, are integers.In [1], Gessel obtained the above congruence by introducing the following method:

Using Taylor's theorem and (1), we have

$$f(x+y) = \sum_{k=0}^{\infty} f^{(k)}(x) y^k / k!, \quad f^{(k)}(x) = \frac{d^k f(x)}{dx^k}. \tag{3}$$

Setting $y = S(z)$ in (3), where the function $S(z)$ is a Hurwitz series in z with $S(0) = 0$ and $S'(0) = 1$ and multiplying both sides by some Hurwitz series $H(z)$ with $H(0) = 1$, we get

$$H(z) f(x + S(z)) = \sum_{k=0}^{\infty} f^{(k)}(x) H(z)(S(z))^k / k!.$$

If the functions $H(z)$ and $S(z)$ are chosen appropriately, the coefficients of $\frac{x^m}{m!} z^n$ on the left will be integral. Then the coefficients of $\frac{x^m}{m!} \frac{z^n}{n!}$ on the right is divisible by $n!$.

In other words, Gessel's method can be applied to a given Hurwitz series $f(x)$ if and only if there exist Hurwitz series $S(z)$ and $H(z)$ with $S(0)=0$, $S'(0)=1$, and $H(0)=1$, such that, for all integers m and n, the coefficients of $\frac{x^m}{m!}z^n$ in $H(z)f(x+S(z))$ is an integer. That is,

$$H(z)f(x+S(z))=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Q(m,n)\frac{x^m}{m!}z^n, \tag{4}$$

where the numbers $Q(m,n)$ are integers for all m and n.

In this paper we establish a necessary and sufficient condition on the function $f(x)$, given by (1), for Gessel's method to be applied, and we show the corresponding congruence concerning the numbers $P_n, n=0,1,2,\dots$. Moreover, we consider a wide class of functions $f(x)$ to which Gessel's method can be applied.

It is well known that Hurwitz series are closed under multiplication and that, if $f(x)$ and $g(x)$ are Hurwitz series with $g(0)=0$, then the composition $(f\circ g)(x)$ is also a Hurwitz series. In particular, $(g(x))^k/k!$ is a Hurwitz series for any nonnegative integer k.

Hurwitz series in two variables are of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\frac{x^m}{m!}\frac{y^n}{n!},$$

where the numbers a_{mn} are integers. The properties of these series we will need follow from those for Hurwitz series in one variable.

We also need the following results:

a. Let $s^{-1}(x)$ be the inverse function of the Hurwitz series $s(x)$ with $s(0)=0$. Then $s^{-1}(x)$ is also a Hurwitz series with $s^{-1}(0)=0$, if $\frac{d}{dx}s(x)\big|_{x=0}=s'(0)=1$.

b. Let $h(x)$ be a Hurwitz series. Then the function $\frac{1}{h(x)}=(h(x))^{-1}$ is a Hurwitz series if and only if $h(0)=1$.

2. THE MAIN RESULTS

A necessary and sufficient condition for Gessel's method to be applied is given by the following theorem.

Theorem 1: Let $f(x)$ be the exponential generating function of the integers $P_n, n=0,1,2,\dots$, as given by (1). Then Gessel's method can be applied to the Hurwitz series $f(x)$ if and only if there exist Hurwitz series $s(y)$ and $h(y)$ with $s(0)=0$, $s'(0)=1$, and $h(0)=1$, such that

$$f(x+y)=h(y)\left[\sum_{n=0}^{\infty}G_n(x)(s(y))^n\right], \tag{5}$$

where the functions $G_n(x), n=0,1,2,\dots$, are Hurwitz series in x.

Proof: From relation (4) we can easily obtain relation (5), setting $z=s(y)$ where s is the inverse function of S, $[H(s(y))]^{-1}=h(y)$ and $\sum_{m=0}^{\infty}Q(m,n)x^m/m!=G_n(x)$

From our comments in section 1, we can easily see that $s(y)$ and $h(y)$ are Hurwitz series in y with $s(0)=0$, $s'(0)=1$, and $h(0)=1$. Conversely, from relation (5) we obtain, in the same way, relation (4).

Example 1: $f(x)=\tan x$ and we have

$$f(x+y)=\tan x+(\sec x)^2\sum_{n=1}^{\infty}(\tan x)^{n-1}(\tan y)^n.$$

Consequently, $h(y)=1$, $G_0(x)=\tan x$, $G_n(x)=\sec^2 x(\tan x)^{n-1}$, $n=1,2,\ldots$, $s(y)=\tan y$, $s^{-1}(z)=\arctan z$, and Theorem 1 can be applied.

Now we show the corresponding congruence concerning the numbers P_n, $n=0,1,2,\ldots$.

From relations (1) and (3), we obtain

$$f(x+y)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}P_{m+k}\frac{x^m}{m!}\frac{y^k}{k!}. \tag{6}$$

Comparing relations (5) and (6), we obtain

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}P_{m+k}\frac{x^m}{m!}\frac{y^k}{k!}=h(y)\left[\sum_{n=0}^{\infty}G_n(x)(s(y))^n\right]. \tag{7}$$

Setting $y=s^{-1}(z)$ in (7) and multiplying both sides by $(h(s^{-1}(z)))^{-1}$, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}P_{m+k}\frac{x^m}{m!}(h(s^{-1}(z)))^{-1}\frac{(s^{-1}(z))^k}{k!}&=\sum_{n=0}^{\infty}G_n(x)z^n\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}n!Q(m,n)\frac{x^m}{m!}\frac{z^n}{n!},\end{aligned} \tag{8}$$

where the integers $Q(m,n)$ are given by the relation

$$\sum_{m=0}^{\infty}Q(m,n)x^m/m!=G_n(x). \tag{9}$$

From our comments in section 1, we can define the integers $D(n,k)$, $k=0,1,\ldots,n$, $n=0,1,2,\ldots$, by

$$\sum_{n=k}^{\infty}D(n,k)z^n/n!=(h(s^{-1}(z)))^{-1}\frac{(s^{-1}(z))^k}{k!}. \tag{10}$$

Substituting (10) into (8) we get, on using the relation $D(0,0)=1$,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}D(n,k)P_{m+k}\right)\frac{x^m}{m!}\frac{z^n}{n!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}n!Q(m,n)\frac{x^m}{m!}\frac{z^n}{n!}.$$

Equating coefficients of $\frac{x^m}{m!}\frac{z^n}{n!}$, we get

$$\sum_{k=0}^{n}D(n,k)P_{m+k}=n!Q(m,n). \tag{11}$$

Now we consider a wide class of Hurwitz series $f(x)$ to which Gessel's method can be applied, by the following theorems.

Theorem 2: Gessel's method can be applied to the Hurwitz series $f(x)$, if

$$f(x) = (1+\beta g(x))^{\alpha} e^{\gamma x}, \tag{12}$$

where the constants $\alpha, \beta,$ and γ are integers and the function $g(x)$ is a Hurwitz series such that

$$g(x+y) = \sum_{n=0}^{\infty} H_n(x)(s(y))^n, \; H_0(0) = 0, \tag{13}$$

where the function $s(y)$ is a Hurwitz series in y with $s'(0) = 1, s(0) = 0,$ and the functions $H_n(x)$, $n = 0, 1, 2, \ldots,$ are Hurwitz series in x.

Proof: From relation (12) we have, on using (13) and some well-known rules of multiplication of series

$$f(x+y) = \left[1+\beta g(x+y)^{\alpha} e^{\gamma(x+y)}\right] = \left[1+\beta H_0(x) + \beta \sum_{n=1}^{\infty} H_n(x)(s(y))^n\right]^{\alpha} e^{\gamma(x+y)}$$

$$= \left[1+\sum_{n=1}^{\infty} H_n^*(x)(s(y))^n\right]^{\alpha} [1+\beta H_0(x)]^{\alpha} e^{\gamma(x+y)},$$

where $H_n^*(x) = \beta H_n(x)/[1+\beta H_0(x)]$ or

$$f(x+y) = [1+\beta H_0(x)]^{\alpha} e^{\gamma(x+y)} \sum_{j=0}^{\infty} \binom{\alpha}{j} \left[\sum_{n=1}^{\infty} H_n^*(x)(s(y))^n\right]^j$$

$$= [1+\beta H_0(x)]^{\alpha} e^{\gamma(x+y)} \left\{1+\sum_{j=1}^{\infty} \binom{\alpha}{j} \left[\sum_{m=j}^{\infty} (s(y))^m \sum H_{n_1}^*(x) H_{n_2}^*(x) \cdots H_{n_j}^*(x)\right]\right\},$$

where the inner sum is extended over all ordered j-tuples $(n_1, n_2, \ldots, n_j)$ of positive integers such that $n_1 + n_2 + \cdots + n_j = m$ or

$$f(x+y) = e^{\gamma y}[1+\beta H_0(x)]^{\alpha} e^{\gamma x} \left\{1+\sum_{m=1}^{\infty} \left[\sum_{j=1}^{m} \binom{\alpha}{j} \sum H_{n_1}^*(x) \cdots H_{n_j}^*(x)\right] (s(y))^m\right\}$$

or

$$f(x+y) = h(y) \sum_{m=0}^{\infty} G_m(x)(s(y))^m, \tag{14}$$

where $h(y) = e^{\gamma y}, G_0(x) = [1+\beta H_0(x)]^{\alpha} e^{\gamma x}$ and

$$G_m(x) = [1+\beta H_0(x)]^{\alpha} e^{\gamma x} \left[\sum_{j=1}^{m} \binom{\alpha}{j} \sum H_{n_1}^*(x) \cdots H_{n_j}^*(x)\right], \; m = 1, 2, \ldots,$$

(the inner sum is extended as before).

Since $G_m(x)$, $m = 0, 1, 2, \ldots$, are Hurwitz series in x and $s(y)$, $h(y)$ are Hurwitz series in y with $s(0) = 0$, $s'(0) = 1$, and $h(0) = 1$, we have, on using relation (14) and Theorem 1, that Gessel's method can be applied.

Example 2: $f(x) = (1+\beta \tan x)^{\alpha}$, α an integer. We have $\gamma = 0$, $g(x) = \tan x$, and

$$g(x+y) = \tan x + (1+\tan^2 x)\sum_{n=1}^{\infty}(\tan x)^{n-1}(\tan y)^n.$$

Consequently, $s(y) = \tan y$, $H_0(x) = \tan x$, $H_n(x) = (1+\tan^2 x)(\tan x)^{n-1}, n = 1, 2, \ldots,$ and Theorem 2 can be applied.

Example 3: $f(x) = e^{\gamma x}(1-\beta(e^x - 1))^{-\alpha}$, where α, β, γ are integers. We have $g(x) = -(e^x - 1)$ and $g(x+y) = -(e^x - 1) - e^x(e^y - 1)$ Hence, $s(y) = e^y - 1$, $H_0(x) = -(e^x - 1)$, $H_1(x) = -e^x$, and Theorem 2 can be applied.

Note that the above function $f(x)$ is the exponential generating function for the moments for the Meixner polynomials.

Theorem 3: Gessel's method can be applied to the Hurwitz series $f(x)$ if

$$f(x) = \exp\{F(x)\}, \tag{15}$$

where $F(x)$ is a Hurwitz series in x such that

$$F(x+y) = L(x) + \sum_{j=0}^{\infty} R_j(y)(r(x))^j / j!, \tag{16}$$

where $L(x)$ is a Hurwitz series in x with $L(0) = 0$, $R_0(y)$ is a Hurwitz series in y with $R_0(0) = 0$, $R_j(y), j = 1, 2, \ldots,$ are power series in $s(y)$ with integer coefficients, $s(y)$ is a Hurwitz series in y with $s(0) = 0$, $s'(0) = 1$, and $r(x)$ is a Hurwitz series in x with $r(0) = 0$.

Proof: Introducing the exponential Bell polyomials $B_n = B_n(b_1, b_2, \ldots, b_n), n = 0, 1, 2, \ldots,$ that may be defined by their exponential generating function as

$$\sum_{n=0}^{\infty} B_n t^n / n! = \exp\{\phi(t)\}$$

where $\phi(t) = \sum_{j=1}^{\infty} b_j t^j / j!$, we get

$$\exp\left\{\sum_{j=1}^{\infty} R_j(y)(r(x))^j / j!\right\} = \sum_{n=0}^{\infty} B_n(R_1(y), \ldots, R_n(y))(r(x))^n / n!. \tag{17}$$

Explicit expressions for $B_n = B_n(b_1, b_2, \ldots, b_n)$ as functions of $b_1, b_2, \ldots, b_n$ are given in Kendall & Stuart ([2], p. 69).

Since $R_j(y), j = 1, 2, \ldots,$ are power series in $s(y)$, we have that $B_n, n = 1, 2, \ldots,$ are also power series in $s(y)$. Therefore,

$$B_n(R_1(y), \ldots, R_n(y)) = \sum_{i=0}^{\infty} \alpha_{n,i}(s(y))^i, \ n = 1, 2, \ldots, \tag{18}$$

where the numbers $\alpha_{n,i}, i = 0, 1, 2, \ldots,$ are integers.

From relation (15) we have, on using relations (18), (17), and (16),

$$f(x+y) = h(y)\sum_{i=0}^{\infty} G_i(x)(s(y))^i, \tag{19}$$

where $h(y) = \exp[R_0(y)]$ and $G_i(x) = \{\exp[L(x)]\}\sum_{n=0}^{\infty} \alpha_{n,i}(r(x))^n / n!,\ \ i = 0, 1, 2, \ldots$. Since $R_0(0) = L(0) = r(0) = 0$, we have that $h(y)$ is a Hurwitz series in y with $h(0) = 1$ and $G_i(x), i = 0, 1, 2, \ldots$, are Hurwitz series in x. We also have $s(0) = 0$ and $s'(0) = 1$. Consequently, using Theorem 1, we conclude that Gessel's method can be applied.

Example 4: $f(x) = \exp\{\sum_{i=1}^{\infty} c_i x^i / i\}, c_i, i = 1, 2, \ldots$, integers. We have $F(x) = \sum_{i=1}^{\infty} c_i x^i / i$ and

$$F(x+y) = \sum_{i=1}^{\infty} c_i (x+y)^i / i = \sum_{i=1}^{\infty} (c_i / i) \sum_{j=0}^{i} \binom{i}{j} x^j y^{i-j}$$

$$= F(x) + F(y) + \sum_{i=2}^{\infty} (c_i / i) \sum_{j=1}^{i-1} \binom{i}{j} x^j y^{i-j} = F(x) + F(y) + \sum_{j=1}^{\infty} R_j(y) x^j / j!,$$

where

$$R_j(y) = \sum_{i=j+1}^{\infty} c_i \frac{(i-1)!}{(i-j)!} y^{i-j} = \sum_{i=1}^{\infty} \left[c_{i+j} \binom{i+j-1}{i} (j-1)! \right] y^i, \ \ j = 1, 2, \ldots .$$

Thus, $L(x) = F(x), R_0(y) = F(y), R_j(y), j = 1, 2, \ldots$, are power series in y with integer coefficients, $s(y) = y, r(x) = x$, and Theorem 3 can be applied.

Note that, for $c_i = 0, i = 3, 4, \ldots$, the above $f(x)$ is the exponential generating function for the moments for the Hermite polynomials.

Example 5: $f(x) = \exp\{\alpha(e^x - 1) - \beta x\}$, α and β integers. We have $F(x) = \alpha(e^x - 1) + \beta x$ and $F(x+y) = \alpha(e^{x+y} - 1) + \beta(x+y) = F(x) + F(y) + (e^y - 1)\alpha(e^x - 1)$. Consequently, $L(x) = F(x)$, $R_0(y) = F(y), R_1(y) = e^y - 1, s(y) = e^y - 1, r(x) = \alpha(e^x - 1)$, and Theorem 3 can be applied.

Note that, for $\beta = 0$, the above $f(x)$ is the generating function for the moments for the Charlier polynomials.

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