GREATEST INTEGER IDENTITIES FOR GENERALIZED FIBONACCI SEQUENCES $\{H_n\}$, WHERE $H_n = H_{n-1} + H_{n-2}$

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The generalized Fibonacci sequence $\{H_n\}$ where $H_n = H_{n-1} + H_{n-2}$, $H_1 = A$, $H_2 = B$, A and B integers, has been studied in the classic paper by Horadam [6] and by Hoggatt [4] and Brousseau [1], among others. Here we develop ten greatest integer identities for $\{H_n\}$. Rather than establishing these identities "for n sufficiently large," we show exact lower boundaries for subscript n dependent upon the subscript of F_k, the k^{th} Fibonacci number.

Let A, B be positive integers with $A \le B$ and define H_n $(= H_n(A, B))$ by

$$H_1 = A, \ H_2 = B, \quad H_n = H_{n-1} + H_{n-2} \text{ for } n \ge 3.$$

It is not difficult to see that in the sequence $B, A, B-A, 2A-B, 2B-3A, 5A-3B, \ldots$ there is a leftmost term the double of which is less than or equal to the preceding term; otherwise, the rational number A/B would satisfy $F_{2n}/F_{2n+1} < A/B < F_{2n+1}/F_{2n+2}$ for all n. Consequently, every sequence $H_n(A, B)$ agrees, except for some initial finite set of terms, with a sequence $H_n(A', B')$, where A' and B' are positive integers with $A' = B'$ or $2A' < B'$. Then, without loss of generality, we take $A = B$ or $2A < B$ to standardize the subscripts of $\{H_n\}$ so that $H_n \ge 0$ for all $n \ge 0$ where we take $H_0 = B - A$. (The term $2A - B$ preceding H_0 will be negative when $A \ne B$.)

For these reasons, in the following we confine our attention to the two cases (a) $A = B$; and (b) $2A < B$. We call these, respectively, the *Fibonacci case* and the *Lucas case.* Throughout, we put $H_0 = B - A$ and define k by $F_{k-1} < H_0 \le F_k$ for $k \ge 3$ in the Lucas case and by $F_k \le A < F_{k+1}$ for $k \ge 2$ in the Fibonacci case.

In the following we prove ten identities for the general sequences $\{H_n\}, 0 < A \le B$. In Sections 2 and 3 we give ten greatest integer properties of $\{H_n\}$ in the Fibonacci and Lucas cases and, finally, in Section 4 we give these ten properties in a form which includes both cases.

1. PROPERTIES OF $\{H_n\}$ WHERE $H_n = H_{n-1} + H_{n-2}$

The following identities needed for our development are true for all $\{H_n\}$, $H_n = H_{n-1} + H_{n-2}$, $0 < A \le B$, and are given in [1], [4], or [6] or else are proved here.

$$H_n = F_{n-2}A + F_{n-1}B. \tag{1.1}$$

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \text{ where } \alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$$
$$\text{are the roots of } x^2 - x - 1 = 0 \text{ and } \alpha\beta = -1, \alpha + \beta = 1 \tag{1.2}$$

$$H_n = c\alpha^n + d\beta^n \text{ for suitable } c \text{ and } d. \tag{1.3}$$

From (1.1) and (1.2),

$$\begin{aligned}\sqrt{5}\,H_n &= A(\alpha^{n-2}-\beta^{n-2})+B(\alpha^{n-1}-\beta^{n-1})\\ &= \alpha^{n-1}(B-\beta A)-\beta^{n-1}(B-\alpha A)\\ &= \alpha^{n}(\beta A-B)\beta+\beta^{n}(B-\alpha A)\alpha\\ &= \alpha^{n}(A-\beta(B-A))+\beta^{n}(\alpha(B-A)-A),\end{aligned}$$

so that one choice for c and d, where $A=H_1$ and $B-A=H_0$, is

$$c=(A-\beta(B-A))/\sqrt{5} \quad\text{and}\quad d=(\alpha(B-A)-A)/\sqrt{5}. \tag{1.4}$$

Identities (1.5) and (1.6) are easily established by mathematical induction:

$$\alpha^{k-2}<F_k<\alpha^{k-1},\ k\ge 3; \tag{1.5}$$

$$1/2^n<|\beta|^n<1/2,\ n\ge 2,\quad |\beta|^n<1/4,\ n\ge 3. \tag{1.6}$$

Lemma 1.7: There exists an expression $K(m)$ such that

$$\alpha^m F_n = F_{n+m}+\beta^{n-m}K(m)$$

where $|K(m)|<1, m\ge 1,$ and $K(m)<0$ if m is even while $K(m)>0$ if m is odd.

Proof: Multiply by α^m in (1.2) to write

$$\alpha^m F_n=\alpha^m(\alpha^n-\beta^n)/\sqrt{5}=(\alpha^{n+m}-\beta^{n+m}+\beta^{n+m}+(-1)^{m+1}\beta^{n-m})/\sqrt{5}$$

$$\alpha^m F_n=(\alpha^{n+m}-\beta^{n+m})/\sqrt{5}+\beta^{n-m}(\beta^{2m}+(-1)^{m+1})/\sqrt{5}, \tag{1.7}$$

which will verify Lemma 1.7. □

Lemma 1.8: There exists an expression $K^*(m), 0<K^*(m)<1$, such that

$$F_n/\alpha^m=F_{n-m}+\beta^{n-m}K^*(m),\ m\ge 1.$$

Proof: Multiplying by $1/\alpha^m$ in (1.2) yields

$$F_n/\alpha^m=(\alpha^n-\beta^n)/\alpha^m\sqrt{5}=(\alpha^{n-m}-\beta^{n-m})/\sqrt{5}+(\beta^{n-m}+(-1)^{m+1}\beta^{n+m})/\sqrt{5}$$

$$F_n/\alpha^m=F_{n-m}+\beta^{n-m}(1+(-1)^{m+1}\beta^{2m})/\sqrt{5}, \tag{1.8}$$

which will verify Lemma 1.8. □

The characteristic number D for $\{H_n\}$ is defined as $D=B^2-AB-A^2$ in [1] and [6], and

$$H_n^2-H_{n-1}H_{n+1}=(-1)^n D, \tag{1.9}$$

where $D>0$ in the Lucas case where $2A<B$, while $D=-1$ for the Fibonacci numbers,

$$F_n^2-F_{n-1}F_{n+1}=(-1)^{n+1}. \tag{1.10}$$

Identities (1.9) and (1.10) show a subtle but important difference in parity between the Lucas and Fibonacci cases, since n even in (1.9) gives a positive value while n even in (1.10) gives a negative value. The difference in parity causes us to consider the Fibonacci and Lucas cases separately.

2. THE FIBONACCI CASE: THE SEQUENCES $\{H_n\}$ WHERE $A = B$

Consider the Fibonacci case for $\{H_n\}$ where $A = B$. Then $H_n = AF_n$, $A \geq 1$. We write ten greatest integer identities which are true for $\{H_n\}$ when $H_n = AF_n$, and hence for $\{F_n\}$, since the Fibonacci sequence is the special case $A = B = 1$. We write $[x]$ to denote the greatest integer contained in x, and in every case, we determine k by $F_k \leq A < F_{k+1}$, $k \geq 2$.

Theorem 2.1: $[\alpha AF_n] = AF_{n+1}$, n odd, $n \geq k, k \geq 2, A \geq 1$;
$[\alpha AF_n] = AF_{n+1} - 1$ n even, $n \geq k, k \geq 2, A \geq 1$.

Proof: Let $m = 1$ in (1.7) to write

$$\alpha F_n = F_{n+1} + \beta^{n-1}(\beta^2 + 1)/\sqrt{5}.$$

Multiplying by A and computing $(\beta^2 + 1)/\sqrt{5} = -\beta$,

$$\alpha AF_n = AF_{n+1} + (-A\beta^n). \tag{2.1}$$

If $A < F_{k+1}$, we have $A < \alpha^k$ by (1.5), and

$$|-A\beta^n| < |\alpha^k \beta^n| = |\beta^{n-k}| < 1$$

for $n \geq k, k \geq 2$, by (1.6). If n is odd, $0 < -A\beta^n < 1$, while if n is even, $0 > -A\beta^n > -1$, giving Theorem 2.1, for $n \geq k, k \geq 2$. □

Theorem 2.2: $[\alpha AF_n + 1/2] = AF_{n+1}$, $n \geq k+2$.

Proof: Since $|-A\beta^n| < |\alpha^k \beta^{n-k}| = |\beta^{n-k}| < 1/2$ if $n \geq k+2$, adding 1/2 to each side of (2.1) will ultimately yield Theorem 2.2. □

Theorem 2.3: $[AF_n / \alpha] = AF_{n-1}$, n odd, $n \geq k, k \geq 2, A \geq 1$;
$[AF_n / \alpha] = AF_{n-1} - 1$, n even, $n \geq k, k \geq 2, A \geq 1$.

Proof: By taking $m = 1$ in (1.8) and multiplying by A,

$$AF_n / \alpha = AF_{n-1} + (-A\beta^n). \tag{2.3}$$

The proof is finished by analyzing $|-A\beta^n|$ as in the proof of Theorem 2.1. □

As in Theorem 2.2, variations of Equation (2.3) will lead to Theorems 2.4 and 2.5; the proofs are omitted.

Theorem 2.4: $[AF_n / \alpha + 1/2] = AF_{n-1}$, $n \geq k+2$.

Theorem 2.5: $[(AF_n + 1)/\alpha] = AF_{n-1}$, $n \geq k+2$.

Theorem 2.6: $[\alpha^m AF_n] = AF_{n+m}$, n odd, $n \geq m+k$;
$[\alpha^m AF_n] = AF_{n+m} - 1$, n even, $n \geq m+k$.

Proof: Multiply by A in Lemma 1.7 to write

$$\alpha^m AF_n = AF_{n+m} + A\beta^{n-m}K(m) \tag{2.6}$$

where $|K(m)| < 1, m \geq 1,$ and $K(m) < 0$ if m is even while $K(m) > 0$ if m is odd. Since also $A < \alpha^k$ when $n \geq m+k$,

$$|A\beta^{n-m}K(m)| < |\alpha^k\beta^{n-m}K(m)| < |\beta^{n-m-k}| < 1.$$

If n is odd, and m even, $K(m) < 0, \beta^{n-m} < 0,$ and $0 < A\beta^{n-m}K(m) < 1,$ while m odd makes the same result from $K(m) > 0$ and $\beta^{n-m} > 0$. Thus, if n is odd, $[\alpha^m AF_n] = AF_{n+m}$..

If n is even, m odd makes $K(m) > 0,\ \beta^{n-m} < 0,$ so that $0 > A\beta^{n-m}K(m) > -1,$ while m even gives the same result from $K(m) < 0$ and $\beta^{n-m} > 0$. If n is even, $[\alpha^m AF_n] = AF_{m+n} - 1$. □

Adding 1/2 to each side of (2.6) will ultimately yield Theorem 2.7.

Theorem 2.7: $[\alpha^m AF_n + 1/2] = AF_{n+m},\ n \geq m+k+2.$

Theorem 2.8 $[AF_n / \alpha^m] = AF_{n-m}, \quad n-m$ even, $n \geq m+k;$

$[AF_n / \alpha^m] = AF_{n-m} - 1, \quad n-m$ odd, $n \geq m+k.$

Proof: Refer to Lemma 1.8 to write

$$AF_n / \alpha^m = AF_{n-m} + A\beta^{n-m}K^*(m) \tag{2.8}$$

where $0 < K^*(m) < 1, m \geq 1,$ and $A < \alpha^k$.

If $n-m$ is even and $n-m \geq k$,

$$0 < A\beta^{n-m}K^*(m) < \alpha^k\beta^{n-m}K^*(m) < 1.$$

If $n-m$ is odd, $\beta^{n-m} < 0$ while $A \geq 1$ and, if $n \geq m$,

$$0 > A\beta^{n-m}K^*(m) > \beta^{n-m}K^*(m) > -|\beta|^{n-m} > -1,$$

finishing the proof. □

Theorem 2.9: $[AF_n / \alpha^m + 1/2] = AF_{n-m},\ n \geq m+k+2.$

Proof: Add 1/2 to each side of (2.8), and analyze the resulting expressions for $n-m$ even, and for $n-m$ odd. □

Theorem 2.10: $AF_n = [A\alpha^n / \sqrt{5} + 1/2],\ n \geq k, k \geq 2, A \geq 1.$

Proof:

$$\begin{aligned} A\alpha^n / \sqrt{5} + 1/2 &= A(\alpha^n / \sqrt{5} - \beta^n / \sqrt{5}) + A\beta^n / \sqrt{5} + 1/2 \\ &= AF_n + A\beta^n / \sqrt{5} + 1/2 \end{aligned}$$

where

$$|A\beta^n / \sqrt{5} + 1/2| < |\alpha^k\beta^n / \sqrt{5} + 1/2| = |\beta^{n-k} / \sqrt{5} + 1/2| < 1$$

for $n \geq k$ and $k \geq 2$. □

If $A=1$, we have, of course, the Fibonacci numbers $\{F_n\}$. Theorems 2.1 and 2.6 for $\{F_n\}$ appear in [5], and Theorems 2.2 and 2.10 in [4], for $A=1$ and $n\geq 2$. By taking $A=1$ in the proof of Theorems 2.2, 2.4, and 2.5, we find that in the special case $AH_n=F_n$ all three are true for $n\geq 2$.

If $\{H_n\}$ contains $H_n=kF_n$ but $H_{n-1}\neq KF_{n-1}$, then we have the Lucas case $A\neq B$ of the next section.

3. THE LUCAS CASE: THE SEQUENCES $\{H_n\}$ WHERE $0<2A<B$

Let $H_n=H_{n-1}+H_{n-2}$ where $H_1=A$, $H_2=B$, and $0<2A<B$. We prove ten greatest integer identities as before, but we define k by

$$F_{k-1}<B-A\leq F_k,\ \ k\geq 3.$$

Referring to (1.5), we can combine inequalities to write

$$B-A<\alpha^{k-1},\ \ k\geq 3;\quad \text{and } 1\leq A. \tag{3.01}$$

By applying (1.4) and (3.01) and making careful analysis of signs, we next establish

$$|\sqrt{5}d\beta^n|<|\beta|^{n-k}-|\beta|^n,\ \ n\geq k, \tag{3.02}$$

where $d=(\alpha(B-A)-A)/\sqrt{5}>0$.

If n is even, $\beta^n>0$, and

$$\begin{aligned}0<\sqrt{5}\,d\beta^n&=(\alpha(B-A)-A)\beta^n<(\alpha\alpha^{k-1}-1)\beta^n\\&=(-1)^k\beta^{n-k}-\beta^n\\&=|\beta|^{n-k}-|\beta|^n.\end{aligned}$$

If n is odd, $-\beta^n>0$, and

$$\begin{aligned}0>\sqrt{5}\,d\beta^n&=(A-\alpha(B-A))(-\beta^n)>(1-\alpha\alpha^{k-1})(-\beta^n)\\&=-\beta^n+(-1)^k\beta^{n-k}\\&=|\beta|^n-|\beta|^{n-k}\end{aligned}$$

which establishes (3.02) and will allow us to write several identities for $\{H_n\}$, in the Lucas case.

Theorem 3.1: $[\alpha H_n]=H_{n+1}$, n even, $n\geq k$;
$[\alpha H_n]=H_{n+1}-1$, n odd, $n\geq k$.

Proof:

$$\begin{aligned}\alpha H_n&=\alpha(c\alpha^n+d\beta^n)\\&=c\alpha^{n+1}+d\beta^{n+1}-d\beta^{n+1}-d\beta^{n-1}\\&=H_{n+1}-d\beta^{n-1}(\beta^2+1)\end{aligned}$$

$$\alpha H_n=H_{n+1}+\sqrt{5}\,d\beta^n. \tag{3.1}$$

By (3.02), $|\sqrt{5}\,d\beta^n|<|\beta|^{n-k}-|\beta|^n<1-1/2^n, n\geq k$, which establishes Theorem 3.1 by considering the cases n even and n odd. □

Theorem 3.2: $[\alpha H_n + 1/2] = H_{n+1},\ n \ge k+2$.

Proof: Add 1/2 to each side of (3.1) and use (3.02) to analyze the result. □

Theorem 3.3: $[H_n/\alpha] = H_{n-1}$, n even, $n \ge k$;
$[H_n/\alpha] = H_{n-1} - 1$, n odd, $n \ge k$.

Proof:

$$H_n/\alpha = (c\alpha^n + d\beta^n)/\alpha = c\alpha^{n-1} + d\beta^{n-1} - d\beta^{n-1} - d\beta^{n+1}$$

$$H_n/\alpha = H_{n-1} + \sqrt{5}\,d\beta^n \tag{3.3}$$

where we note the same fractional expression $\sqrt{5}\,d\beta^n$ as in Theorem 3.1. □

Theorem 3.3 corrects a proof of a theorem of Cohn [2; p. 31], in which he gives the next lower term to N as $[N/\alpha]$, which is true when n is even but not when n is odd. Dr. Cohn has acknowledged the error in a private correspondence with one of the authors.

Theorem 3.4: $[H_n/\alpha + 1/2] = H_{n-1},\ n \ge k+2$.

The proof is identical to that of Theorem 3.2, but using (3.3). □

Theorem 3.5: $[(H_n+1)/\alpha] = H_{n-1},\ n \ge k+3$.

Proof: From (3.3),

$$(H_n+1)/\alpha = H_{n-1} + \sqrt{5}\,d\beta^n + 1/\alpha.$$

By (3.02), $|\sqrt{5}\,d\beta^n| < 1/4 - 1/2^n$ for $n \ge k+3$. Adding $1/\alpha$ to each term of the inequality for the case n even, and then for the case n odd, we find that in either case, we obtain $0 \le \sqrt{5}\,d\beta^n + 1/\alpha < 1$. □

Theorem 3.6: $[\alpha^m H_n] = H_{n+m}$, n even, $n \ge m+k, m \ge 2$;
$[\alpha^m H_n] = H_{n+m} - 1$, n odd, $n \ge m+k, m \ge 2$.

Proof: Since $1/\alpha^m = (-1)^m \beta^m$,

$$\begin{aligned}\alpha^m H_n &= \alpha^m(c\alpha^n + d\beta^n)\\ &= c\alpha^{n+m} + d\beta^{n+m} - d\beta^{n+m} + (-1)^m d\beta^{n-m}\\ &= H_{n+m} + \sqrt{5}\,d\beta^{n-m}((-1)^m - \beta^{2m})/\sqrt{5}\end{aligned}$$

$$\alpha^m H_n = H_{n+m} + \sqrt{5}\,d\beta^{n-m} M(m) \tag{3.6}$$

where $|M(m)| < 1$ for $m \ge 1$. By (3.02),

$$|\sqrt{5}\,d\beta^{n-m} M(m)| \le |\sqrt{5}\,d\beta^{n-m}| < |\beta|^{n-m-k} - |\beta|^{n-m} < 1 - 1/2^{n-m}$$

for $n - m \ge k$. Consider the signs carefully. For n odd, m odd, $\beta^{n-m} > 0$ and $M(m) < 0$, while for n odd, m even, $\beta^{n-m} < 0$ and $M(m) > 0$, so whenever n is odd,

$$0 > \sqrt{5}\,d\beta^{n-m} M(m) > 1/2^{n-m} - 1 > -1,$$

so that $[\alpha^m H_n] = H_{n+m} - 1$. For n even, m odd, $\beta^{n-m} < 0$ and $M(m) < 0$, while for n even, m even, $\beta^{n-m} > 0$ and $M(m) > 0$, so whenever n is even,

$$0 < \sqrt{5}\, d\beta^{n-m} M(m) < 1 - 1/2^{n-m} < 1,$$

so that $[\alpha^m H_n] = H_{n+m}$. □

Theorem 3.7: $[\alpha^m H_n + 1/2] = H_{n+m}$, $n \ge m + k + 2$.

Proof: By (3.6),

$$\alpha^m H_n + 1/2 = H_{n+m} + \sqrt{5}\, d\beta^{n-m} M(m) + 1/2$$

where $|\sqrt{5}\, d\beta^{n-m} M(m)| < 1/2 - 1/2^{n-m}$ for $n - m \ge k + 2$. Add 1/2 to each member of the inequalities for the even and odd cases as in the proof of Theorem 3.2. □

Theorem 3.8:
$$[H_n / \alpha^m] = H_{n-m}, \qquad n - m \text{ odd}, \quad n > m, n - m \ge k;$$
$$[H_n / \alpha^m] = H_{n-m} - 1, \qquad n - m \text{ even}, \quad n > m, n - m \ge k.$$

Proof: Since $1/\alpha^m = (-1)^m \beta^m$,

$$\begin{aligned} H_n / \alpha^m &= (c\alpha^n + d\beta^n) / \alpha^m \\ &= c\alpha^{n-m} + d\beta^{n-m} - d\beta^{n-m} + d(-1)^m \beta^{n+m} \\ &= H_{n-m} + d\beta^{n-m}(-1 + (-1)^{m+1}\beta^{2m}) \\ &= H_{n-m} + \sqrt{5}\, d\beta^{n-m}((-1 + (-1)^{m+1}\beta^{2m})\sqrt{5}) \end{aligned}$$

$$H_n / \alpha^m = H_{n-m} + \sqrt{5}\, d\beta^{n-m} J(m) \tag{3.8}$$

where $|J(m)| < 1$ for $m \ge 1$ but $J(m) < 0$ for $m \ge 1$. From (3.02) we have the same results as in the proof of Theorem 3.6 except for the signs:

$$|\sqrt{5}\, d\beta^{n-m} J(m)| < |\beta|^{n-m-k} - |\beta|^{n-m}.$$

For n odd and m odd, or for n even and m even, $\beta^{n-m} > 0$ and $J(m) < 0$, and we have

$$0 > \sqrt{5}\, d\beta^{n-m} J(m) > 1/2^{n-m} - 1 > -1$$

for $n - m \ge k, n - m$ even, making $|H_n / \alpha^m] = H_{n-m} - 1$.

For n even and m odd, or for n odd and m even, $\beta^{n-m} < 0$ and $J(m) < 0$;

$$0 < \sqrt{5}\, d\beta^{n-m} J(m) < 1 - 1/2^{n-m} < 1$$

for $n - m$ odd, $n - m \ge k$, and $[H_n / \alpha^m] = H_{n-m}$, finishing the proof. □

Theorem 3.9: $[H_n / \alpha^m + 1/2] = H_{n-m}, n > m, n - m \ge k + 2$.

Theorem 3.9 is proved by using the methods of Theorems 3.2 and 3.7 to operate on (3.8).

Theorem 3.10: $H_n = [c\alpha^n + 1/2]$, $n \ge k$, where $c = (H_1 - \beta H_0)/\sqrt{5}$.

Proof: By (1.3) and (1.4),

$$c\alpha^n + 1/2 = c\alpha^n + d\beta^n - d\beta^n + 1/2 = H_n - d\beta^n + 1/2.$$

Divide each term of inequality (3.2) by $\sqrt{5}$ to write

$$|d\beta^n| < (|\beta|^{n-k} - |\beta|^n)/\sqrt{5} < (1 - 1/2^n)/\sqrt{5} < 1/2, \ n \ge k.$$

If n is even, then $\beta^n > 0$, and $0 > d\beta^n > -1/2$. Add 1/2 to each term to determine that $1 > 1/2 - d\beta^n > 0$. If n is odd, then $\beta^n < 0$, and $0 < -d\beta^n < 1/2$ gives $0 < 1/2 - d\beta^n < 1$ upon adding 1/2 to each term. In either case, $H_n = [c\alpha^n + 1/2]$. □

Corollary 3.10: $L_n = [\alpha^n + 1/2]$ for the Lucas numbers $(L_n), n \ge 2$.

Corollary 3.10 appears in [4].

4. THE GENERAL CASE: (H_n) WHERE A = B OR 0 < 2A < B

In comparing the ten theorems of Sections 2 and 3, notice close agreement except for whether subscripts are odd or even, as expected from (1.9) and (1.10). The following results are true for both the Fibonacci and Lucas cases, and hence for all $\{H_n\}$, where we take k from $F_{k-1} < H_0 = B - A < F_{k+1}$ if $A \ne B$, and from $F_k \le A < F_{k+1}$ if $A = B$.

Theorem 4.1: $[\alpha H_n] = H_{n+1}$ or $H_{n+1} - 1, \ n \ge k$.

Theorem 4.2: $[\alpha H_n + 1/2] = H_{n+1}, \ n \ge k+2$.

Theorem 4.3: $[H_n/\alpha] = H_{n-1}$ or $H_{n-1} - 1, \ n \ge k$.

Theorem 4.4: $[H_n/\alpha + 1/2] = H_{n-1}, \ n \ge k+2$.

Theorem 4.5: $[(H_n + 1)/\alpha] = H_{n-1}, \ n \ge k+3$.

Theorem 4.6: $[\alpha^m H_n] = H_{n+m}$ or $H_{n+m} - 1, \ n \ge m+k+2, m \ge 2$.

Theorem 4.7: $[\alpha^m H_n + 1/2] = H_{n+m}, \ n \ge m+k+2, m \ge 2$.

Theorem 4.8 $[H_n/\alpha^m] = H_{n-m}$ or $H_{n-m} - 1, \ n \ge m+k, m \ge 2$.

Theorem 4.9: $[H_n/\alpha^m + 1/2] = H_{n-m}, \ n \ge m+k+2, m \ge 2$.

Theorem 4.10: $[c\alpha^n + 1/2] = H_n, \ c = (H_1 - \beta H_0)/\sqrt{5}, \ n \ge k$.

We can extend Theorems 4.1 through 4.10 for negative subscripts. Since $(-1)^{n+1}F_{-n} = F_n$, $|F_{-n}| = F_n$, Theorems 2.1 through 2.10 apply for sequences having $|F_{-n}|$ or $|AF_{-n}|$ as the n^{th} term. We can apply Theorems 3.1 through 3.10 for $\{H^*_n\}$ where $H^*_n = |H_{-n}|$ as well if we extend the definition of $\{H_n\}$ for negative subscripts so that (1.1) becomes

$$H_{-n} = AF_{-n-2} + BF_{-n-1} = A(-1)^{n+3}F_{n+2} + B(-1)^{n+2}F_{n+1},$$

$$H_{-n} = (-1)^n(BF_{n+1} - AF_{n+2}) = (-1)^n H^*_n, \tag{4.1}$$

where $\{H^*_n\}$ is the conjugate sequence [3] for $\{H_n\}$, $H^*_n = H^*_{n-1} + H^*_{n-2}$, $H^*_0 = B - A$, $H^*_1 = B - 2A = A^*$, $H^*_2 = 2B - 3A = B^*$. Notice that $|H_{-n}| = (-A^*)F_{n-2} + B^*F_{n-1} = H^*_n$, where $\{H^*_n\}$ is one of the sequences $\{H_n\}$ with positive subscripts. Thus, Theorems 3.1 through 3.10 and 4.1 through 4.10 can be extended to $\{H_n\}$ with negative subscripts by taking $|H_n| = H^*_n$ in all the theorem statements.

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REFERENCES

1. Brother Alfred Brousseau. *Introduction to Fibonacci Discovery*, pp. 19-25. Santa Clara, Calif.: The Fibonacci Association, 1965.
2. John H. E. Cohn. "Recurrent Sequences Including *N*." *The Fibonacci Quarterly* **29.1** (1991):30-36.
3. David A. Englund. "Entry Point Reciprocity of Characteristic Conjugate Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **29.3** (1991):197-99.
4. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*, pp. 30-36. Rpt. Santa Clara, Calif.: The Fibonacci Association, 1972.
5. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio." *The Fibonacci Quarterly* **17.4** (1979):306-18.
6. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* **68.5** (1961): 455-59.
7. J. E. Walton & A. F. Horadam. "Some Aspects of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **12.3** (1974):241-50.
8. J. E. Walton & A. F. Horadam. "Some Further Identities for the Generalized Fibonacci Sequence $\{H_n\}$." *The Fibonacci Quarterly* **12.3** (1974):272-80.

Additional references for this paper were found upon reading the August 1994 issue of this quarterly, where Wayne L. McDaniel ("On the Greatest Integer Function and Lucas Sequences" **32.4** [1994]:297-300) gives related but not identical results to those appearing in this paper. In earlier issues of *The Fibonacci Quarterly*, Robert Anaya and Janice Crump ("A Generalized Greatest Integer Function Theorem" **10.2** [1972]:207-12) proved a special case of our Theorem 2.7, and L. Carlitz ("A Conjecture Concerning Lucas Numbers" **10.5** [1972]:526) proved a special case of our Theorem 3.7.

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