ON THE (3, *F*) GENERALIZATIONS OF THE FIBONACCI SEQUENCE

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Recently the (2, *F*) generalized Fibonacci sequences were derived (see [1], [4], [5], [6]). The purpose of this paper is to derive formulas for all (3, *F*) generalized Fibonacci sequences as functions of the terms of seven sixth-order recursive sequences.

Let $S=\{a,b,c\}$ and S_c be the group of permutations on S. Let ι be the identity; and $\alpha=(a\ b)$, $\beta=(a\ c)$, $\gamma=(b\ c)$ be the two cycles; and $\delta=(a\ b\ c)$, $\varepsilon=(a\ c\ b)$ be the three cycles. Finally, let ϕ and τ be arbitrary permutations of S_c and $Y_1=\{a_i,b_i,c_i\}$. Atanassov [2] considered the 36 possible systems of three second-order difference equations:

$$\iota Y_{n+2}=\phi Y_{n+1}+\tau Y_n,\quad n\geq 0, \tag{1}$$

with initial conditions $Y_0=\{a_0,b_0,c_0\}$ and $Y_1=\{a_1,b_1,c_1\}$, where $a_0,b_0,c_0,a_1,b_1,$ and c_1 are real numbers. Since the permutation in the left member of (1) is always the identity (ι), these systems can be represented by the ordered pair (ϕ,τ), where ϕ and τ are the permutations of the right member. Spickerman et al. [7] proved that the 36 systems are members of the eleven equivalence classes. The solutions of one of these classes is three generalized Fibonacci sequences. The solutions to three other classes consist of one generalized Fibonacci sequence and one (2, *F*) generalized Fibonacci sequence. The solutions to the other seven systems are the (3, *F*) generalized Fibonacci sequences. Atanassov [3] denoted each of these seven sequences by a number, as shown in Table 1. A notation in terms of ordered pairs of permutations of S_c is also given.

Considering each equivalence class, it follows that when the solution to one system in a class is known, the solutions to the other systems are permutations of the known solution. Atanassov et al. [3] proved

$$\iota Y_{n+2}^s=\phi Y_{n+1}^s+\tau Y_n^s,\quad s\in\{1,2,3,4,5,6,7\}, \tag{2}$$

with initial conditions

$$Y_0^s=\{a_0^s,b_0^s,c_0^s\},\quad Y_1=\{a_1^s,b_1^s,c_1^s\},$$

can be replaced with seven sixth-order difference systems:

$$\sum_{i=0}^{6}k_i^s a_{n+6-i}^s=0,\quad \sum_{i=0}^{6}k_i^s b_{n+6-i}^s=0,\quad \sum_{i=0}^{6}k_i^s c_{n+6-i}^s=0,\quad n\geq 0, \tag{3}$$

with initial conditions $\{a_i^s\}_0^5$, $\{b_i^s\}_0^5$, $\{c_i^s\}_0^5$, respectively. The values for k_i^s for $1\leq s\leq 7$ are given in Table 2.

Let $p^s(x)=\sum_{i=0}^6 k_i^s x^i$ and let $\{P_j^s\}_{j=0}^\infty$ be the recursive numbers (of order six) determined by $1/p^s(x)$. Then the seven recursion relations and first terms of the sequences are given in Table 3.

TABLE 1

Permutation Notation	Equivalence Class	Atanassov's Number
$[(\iota, \iota)]$	$\{(\iota, \iota)\}$	none*
$[(\alpha, \alpha)]$	$\{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\}$	none**
$[(\iota, \alpha)]$	$\{(\iota, \alpha), (\iota, \beta), (\iota, \gamma)\}$	none**
$[(\alpha, \iota)]$	$\{(\alpha, \iota), (\beta, \iota), (\gamma, \iota)\}$	none**
$[\iota, \delta]$	$\{(\iota, \delta), (\iota, \varepsilon)\}$	1
$[\delta, \iota]$	$\{(\delta, \iota), (\varepsilon, \iota)\}$	2
$[\delta, \delta]$	$\{(\delta, \delta), (\varepsilon, \varepsilon)\}$	3
$[\delta, \varepsilon]$	$\{(\delta, \varepsilon), (\varepsilon, \delta)\}$	4
$[\alpha, \beta]$	$\{(\alpha, \beta), (\alpha, \gamma), (\beta, \alpha), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta)\}$	5
$[\alpha, \delta]$	$\{(\alpha, \delta), (\alpha, \varepsilon), (\beta, \delta), (\beta, \varepsilon), (\gamma, \delta), (\gamma, \varepsilon)\}$	6
$[\delta, \alpha]$	$\{(\delta, \alpha), (\varepsilon, \alpha), (\delta, \beta), (\varepsilon, \beta), (\delta, \gamma), (\varepsilon, \gamma)\}$	7

*Solution is three generalized Fibonacci sequences.
**Solution is one generalized Fibonacci sequence and one (2, F) generalized Fibonacci sequence.

TABLE 2

	Values of k_i^S						
				i			
s	0	1	2	3	4	5	6
1	1	−3	3	−1	0	0	−1
2	1	0	−3	−1	3	0	−1
3	1	0	0	−1	−3	−3	−1
4	1	0	0	−4	0	0	−1
5	1	−1	−2	2	−1	0	1
6	1	−1	−1	0	1	−1	−1
7	1	0	−1	−2	−2	1	1

TABLE 3

s	Recursive Relation	First 7 Terms
1	$P_{n+6} = 3P_{n+5} - 3P_{n+4} + P_{n+3} + P_n$	1, 3, 6, 10, 15, 21, 29
2	$P_{n+6} = 3P_{n+4} + P_{n+3} - 3P_{n+2} + P_n$	1, 0, 3, 1, 6, 6, 11
3	$P_{n+6} = P_{n+3} + 3P_{n+2} + 3P_{n+1} + P_n$	1, 0, 0, 1, 3, 3, 2
4	$P_{n+6} = 4P_{n+3} + P_n$	1, 0, 0, 4, 0, 0, 17
5	$P_{n+6} = P_{n+5} + 2P_{n+4} - 2P_{n+3} + P_{n+2} - P_n$	1, 1, 3, 3, 8, 9, 21
6	$P_{n+6} = P_{n+5} + P_{n+4} - P_{n+2} + P_{n+1} + P_n$	1, 1, 2, 3, 4, 7, 11
7	$P_{n+6} = P_{n+4} + 2P_{n+3} + 2P_{n+2} - P_{n+1} - P_n$	1, 0, 1, 2, 3, 3, 8

Let $f^s(x)$, $g^s(x)$, and $h^s(x)$ be the three solutions to the seven systems, and let

$$f^s(x) = \sum_{j=0}^{\infty} a_j^s x^i, \quad g^s(x) = \sum_{j=0}^{\infty} b_j^s x^i, \quad h^s(x) = \sum_{j=0}^{\infty} c_j^s x^j.$$

First, it follows that

$$\left(\sum_{i=0}^{6} k_i^s x^i\right) f^s(x) = \left(\sum_{i=0}^{6} k_i^s x^i\right)\left(\sum_{j=0}^{\infty} a_j^s x^j\right) = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{n} k_i^s a_{j-i}^s\right] x^j \quad \begin{cases} n = j & \text{for } j \le 5, \\ n = 6 & \text{otherwise,} \end{cases}$$

$$= \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j + \sum_{j=6}^{\infty}\left[\sum_{i=0}^{6} k_i^s a_{j-i}^s\right] x^j.$$

In view of the difference systems (3), the last term is zero. Therefore,

$$\left(\sum_{i=0}^{6} k_i^s x^i\right) f^s(x) = \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j,$$

or

$$p^s(x) f^s(x) = \sum_{j=0}^{5}\left[\sum_{i=0}^{j} k_i^s a_{j-i}^s\right] x^j.$$

Let

$$q_j^s = \sum_{m=0}^{j} k_m^s a_{j-m}^s,$$

then

$$f^s(x) = \frac{\sum_{i=0}^{5} q_i^s x^i}{p^s(x)} = \left(\sum_{i=0}^{5} q_i^s x^i\right)\left(\frac{1}{p^s(x)}\right).$$

Consequently,

$$f^s(x) = \left(\sum_{i=0}^{5} q_i^s x^i\right)\left(\sum_{j=0}^{\infty} P_j^s x^j\right),$$

where P_j^s are from the sequences in Table 2. Expanding and collecting terms gives

$$f^s(x) = \sum_{j=0}^{\infty}\left[\sum_{i=0}^{m} q_i^s P_{j-i}^s\right] x^j \quad \begin{cases} m = j & \text{when } j < 5, \\ m = 5 & \text{otherwise,} \end{cases}$$

$$= \sum_{j=0}^{4}\left(\sum_{i=0}^{j} q_i^s P_{j-i}^s\right) x^j + \sum_{j=0}^{\infty}\sum_{i=0}^{5}\left(q_i^s P_{j-i}^s\right) x^j$$

for the generating function for $\{a_i^s\}_0^\infty$. The terms of the sequence are given by

$$a_j^s = \sum_{i=0}^{j} q_j^s P_{j-i}^s = \sum_{i=0}^{j}\left[\sum_{m=0}^{i} k_m^s a_{i-m}^s\right] P_{j-i}^s \qquad \text{for } j < 5,$$

and

$$a_j^s = \sum_{i=0}^{5} q_i^s P_{j-i}^s = \sum_{i=0}^{5}\left[\sum_{m=0}^{i} k_m^s a_{i-m}^s\right] P_{j-i}^s \qquad \text{for } j \ge 5.$$

The values of $a_i^s, 2 \le i \le 5$, are computed in terms of $a_0^s, a_1^s, b_0^s, b_1^s, c_0^s, c_1^s$ by use of equations (2). The sequences $\{b_i^s\}_0^\infty$ and $\{c_i^s\}_0^\infty$ have the same form.

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