DYNAMICS OF THE MAPPING $f(x) = (x+1)^{-1}$

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(*Submitted February 1994*)

A concise presentation of the dynamics of the sequences $x_{n+1} = f(x_n)$ generated by the function $f(x) = (x+1)^{-1}$ is given. This sequence has a limit for almost all real initial x_0.

To prove this, notice that the interval $(-\infty, 0]$ is eventually mapped onto the positive real axis with the exception of a negative fixed point in $[-2, -1)$, as well as a countable set of points in this interval which are mapped to -1 after n iterations. To obtain this discrete set of initial x_0, one solves the simple equation $f(x_0) = -1$ then inductively, using the fundamental recursion for the Fibonacci sequence, this set of x_0 is given by $S = \{v_n\}$, where

$$v_n = -\frac{F_{n+2}}{F_{n+1}}, \quad n \geq 1.$$

Since $(F_n / F_{n-1}) \to r$, where $r = (1+\sqrt{5})/2$ as $n \to \infty$ (see [1]), it is interesting to note that the sequence $v_n \to -r$ as $n \to \infty$, where $-r$ is the negative fixed point of f.

The behavior of the sequence v_n on each side of $-r$ is somewhat more complicated, but interesting. If $-\infty < x_0 < -2$, then $0^- > (x_0+1)^{-1} > -1$, and if $0 > x > -1$, then $1 < (x+1)^{-1} < \infty$ so that $(-\infty, -2) \to (-1, 0) \to [1, \infty)$.

The most complicated dynamics is on the set $(-2, -1)$ which contains S as a subset. In general, for $n > 2$ such that $v_n > v_{n+2} > -r$, then under the action of f,

$$\left(-\frac{F_{n+4}}{F_{n+3}}, -\frac{F_{n+2}}{F_{n+1}}\right) \to \left(-\frac{F_{n+1}}{F_n}, -\frac{F_{n+3}}{F_{n+2}}\right)$$

and for $n > 2$ such that $-r > v_{n+2} > v_n$, the order of the endpoints is reversed. Therefore, each interval of this form is mapped onto a corresponding interval which is on the opposite side of the fixed point $-r$. and for any $x_0 \in (-2, -1)$ such that $x_0 \neq -r$ or $x_n \notin S$, there is an N such that x_N leaves $(-2, -1)$ and is contained in an interval which has been considered. Convergence can be forced on $x_0 \in S$ if we set $f(\pm\infty) = 0$, then there remains only the unstable fixed point $-r$ which remains invariant.

To finish the study of the convergence of f under iteration, this shows that it suffices to consider $x_N = x_0 \in (0, \infty)$, and since $[1, \infty)$ is mapped onto $(0, 1]$, suppose $x_0 \in (0, 1]$. An expression which generates the x_n and which converges to the positive stable fixed point of the mapping can be written as

$$x_n = \frac{F_{n-1}x_0 + F_n}{F_n x_0 + F_{n+1}}, \quad x_0 \in (0, 1].$$

By induction on n, one obtains

$$x_{n+1} = f(x_n) = \frac{F_n x_0 + F_{n+1}}{F_{n-1}x_0 + F_n + F_n x_0 + F_{n+1}} = \frac{F_n x_0 + F_{n+1}}{F_{n+1}x_0 + F_{n+2}},$$

and then letting $n \to \infty$ one obtains

$$x_\infty = \frac{1}{r} = \frac{\sqrt{5}-1}{2}.$$

REFERENCE

1. I. Niven & H. Zuckermann. *An Introduction to the Theory of Numbers*, 1972.

AMS Classification Numbers: 11B39, 11A99

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