A CONGRUENCE FOR FIBONOMIAL COEFFICIENTS MODULO p^3

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An interesting property of binomial coefficients is that, for primes $p>3$,

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^k} \tag{1}$$

for $k = 1, 2, 3$.

The *Fibonomial coefficients*, defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n F_{n-1} \dots F_1}{(F_k F_{k-1} \dots F_1)(F_{n-k} \dots F_1)},$$

or, more generally,

$$\begin{bmatrix} n \\ k \end{bmatrix}_j = \frac{F_{nj} F_{(n-1)j} \dots F_j}{(F_{kj} F_{(k-1)j} \dots F_j)(F_{(n-k)j} \dots F_j)},$$

where F_i is the i^{th} Fibonacci number. Such expressions have been shown to possess many properties similar to binomial coefficients. In a previous paper [5] the authors investigated properties of Fibonomial coefficients similar to the property (1) of binomial coefficients for $k=2$. The main results of that paper are:

$$\begin{bmatrix} ra \\ rb \end{bmatrix} \equiv \varepsilon^{(a-b)br} \binom{a}{b} \pmod{p^2} \tag{2}$$

and

$$\begin{bmatrix} \tau a \\ \tau b \end{bmatrix} \equiv \binom{ta}{tb} \pmod{p^2}, \tag{3}$$

where τ is the period of the Fibonacci sequence modulo an odd prime p, r is the rank of apparition of p (that is, F_r is the first nonzero F_i divisible by p), and $t = \tau / r$ is an integer. In [7] it is shown that t must assume the value 1, 2, or 4. The number ε is defined by $\varepsilon = 1$ if $\tau = r$, $\varepsilon = -1$ if $\tau = 2r$, and $\varepsilon^2 \equiv -1 \pmod{p^2}$ if $\tau = 4r$.

Unlike the ordinary binomial coefficients, these results are not true in general for higher powers of p. However, in some cases they can be extended to congruences modulo p^3.

In order to prove these results, we will first examine some congruences involving certain products of consecutive Fibonacci numbers. Throughout the paper, L_i represents the i^{th} Lucas number, and $p>3$ is prime.

We first consider $\prod_{k=1}^{r-1} F_{mr+k}$ modulo p^3. From the identity $2F_{a+b} = L_a F_b + L_b F_a$, we obtain $2F_{mr+k} = L_{mr} F_k + L_k F_{mr}$ so that, upon expanding the product and using the facts $p | F_r$ and $F_r | F_{mr}$, we have $p | F_{mr}$ and

$$2^{r-1} \prod_{k=1}^{r-1} F_{mr+k} \equiv (L_{mr}^{r-1} + L_{mr}^{r-2} F_{mr} \Sigma_1 + L_{mr}^{r-3} F_{mr}^2 \Sigma_2) \left(\prod_{k=1}^{r-1} F_k \right) \pmod{p^3}, \tag{4}$$

where

$$\Sigma_1 = \sum_{k=1}^{r-1} \frac{L_k}{F_k} \quad \text{and} \quad \Sigma_2 = \sum_{\substack{n,k=1 \\ k<n}}^{r-1} \frac{L_k}{F_k} \frac{L_n}{F_n}.$$

Then, upon dividing both sides of (4) by $(2^{r-1})\prod_{k=1}^{r-1} F_k$,

$$\frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k} \equiv \left(\frac{L_{mr}}{2}\right)^{r-1} + \frac{1}{2}\left(\frac{L_{mr}}{2}\right)^{r-2} F_{mr}\Sigma_1 + \frac{1}{4}\left(\frac{L_{mr}}{2}\right)^{r-3} F_{mr}^2 \Sigma_2$$
$$\equiv \frac{1}{4}\left(\frac{L_{mr}}{2}\right)^{r-3} [L_{mr}^2 + L_{mr}F_{mr}\Sigma_1 + F_{mr}^2\Sigma_2] \pmod{p^3}. \tag{5}$$

We will next work toward simplifying the right-hand side of (5), specifically we will eliminate Σ_2 by writing it in terms of Σ_1.

Now, because $\Sigma_1 = \sum_{k=1}^{r-1}(L_k / F_k)$, we see that

$$\Sigma_1^2 = \left(\sum_{k=1}^{r-1} \frac{L_k}{F_k}\right)^2 = \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 + 2\sum_{\substack{n,k=1 \\ k<n}}^{r-1} \frac{L_k}{F_k}\frac{L_n}{F_n} = \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 + 2\Sigma_2,$$

thus

$$\Sigma_2 = \frac{1}{2}\left[\Sigma_1^2 - \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2\right]. \tag{6}$$

Now $\Sigma_1 \equiv 0 \pmod p$ [5] so that, from (6), we obtain

$$F_{mr}^2\Sigma_2 \equiv -\frac{1}{2}F_{mr}^2 \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 \pmod{p^4}. \tag{7}$$

We look at $\sum_{k=1}^{r-1}(L_k / F_k)^2$ modulo p^2. Clearly,

$$2\sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 = \sum_{k=1}^{r-1}\left[\left(\frac{L_k}{F_k}\right)^2 + \left(\frac{L_{r-k}}{F_{r-k}}\right)^2\right] = \sum_{k=1}^{r-1}\left[\frac{(L_kF_{r-k})^2 + (L_{r-k}F_k)^2}{(F_kF_{r-k})^2}\right],$$

and, from an identity already mentioned,

$$(2F_r)^2 = (L_kF_{r-k} + L_{r-k}F_k)^2 = (L_kF_{r-k})^2 + (L_{r-k}F_k)^2 + 2(L_kF_{r-k}L_{r-k}F_k),$$

which implies

$$(L_kF_{r-k})^2 + (L_{r-k}F_r)^2 \equiv -2(L_kF_{r-k}L_{r-k}F_k) \pmod{p^2}.$$

Then, substituting in the equality just above,

$$2\sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 = \sum_{k=1}^{r-1}\left[\frac{(L_kF_{r-k})^2 + (L_{r-k}F_k)^2}{(F_kF_{r-k})^2}\right] \equiv \sum_{k=1}^{r-1}\frac{-2(L_kF_{r-k}L_{r-k}F_k)}{(F_kF_{r-k})^2} \equiv -2\sum_{k=1}^{r-1}\frac{L_k}{F_k}\frac{L_{r-k}}{F_{r-k}} \pmod{p^2}$$

or

$$\sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 \equiv -\sum_{k=1}^{r-1}\frac{L_k}{F_k}\frac{L_{r-k}}{F_{r-k}} \pmod{p^2}.$$

We now use the identity $2L_{a+b} = L_aL_b + 5F_aF_b$ to note that $2L_r = L_kL_{r-k} + 5F_kF_{r-k}$; hence,

$$5 + \frac{L_kL_{r-k}}{F_kF_{r-k}} = \frac{2L_r}{F_kF_{r-k}}.$$

Thus,

$$-\sum_{k=1}^{r-1}\frac{L_k}{F_k}\frac{L_{r-k}}{F_{r-k}} = \sum_{k=1}^{r-1}\left(5 - \frac{2L_r}{F_kF_{r-k}}\right) = 5(r-1) - \sum_{k=1}^{r-1}\frac{2L_r}{F_kF_{r-k}}$$

or

$$\sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 \equiv 5(r-1) - \sum_{k=1}^{r-1}\frac{2L_r}{F_kF_{r-k}} \pmod{p^2}. \tag{8}$$

Then, from (7) and (8), we have

$$F_{mr}^2\Sigma_2 \equiv \frac{-5}{2}F_{mr}^2(r-1) + F_{mr}^2\sum_{k=1}^{r-1}\frac{L_r}{F_kF_{r-k}} \pmod{p^4}$$

or

$$F_{mr}^2\Sigma_2 \equiv \frac{-5}{2}F_{mr}^2(r-1) + L_r\frac{F_{mr}^2}{F_r}\sum_{k=1}^{r-1}\frac{F_r}{F_kF_{r-k}} \pmod{p^4}.$$

However,

$$2\Sigma_1 = \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k} + \frac{L_{r-k}}{F_{r-k}}\right) = \sum_{k=1}^{r-1}\frac{L_kF_{r-k} + L_{r-k}F_k}{F_kF_{r-k}} = \sum_{k=1}^{r-1}\frac{2F_r}{F_kF_{r-k}}$$

so that

$$\Sigma_1 = \sum_{k=1}^{r-1}\frac{F_r}{F_kF_{r-k}}. \tag{9}$$

Hence, from the last congruence,

$$F_{mr}^2\Sigma_2 \equiv \frac{-5}{2}F_{mr}^2(r-1) + L_r\frac{F_{mr}^2}{F_r}\Sigma_1 \pmod{p^4},$$

and so, substituting into (5),

$$\frac{\prod_{k=1}^{r-1}F_{mr+k}}{\prod_{k=1}^{r-1}F_k} \equiv \left(\frac{L_{mr}}{2}\right)^{r-1} - \frac{5}{8}F_{mr}^2\left(\frac{L_{mr}}{2}\right)^{r-3}(r-1) + \left(\frac{L_{mr}}{2}\right)^{r-2}\frac{1}{2}\left(F_{mr} + \frac{L_r}{L_{mr}}\frac{F_{mr}^2}{F_r}\right)\Sigma_1 \pmod{p^3}. \tag{10}$$

It is known that, for $p \neq 5$, r divides either $p-1$ or $p+1$, so we will look at the two special cases where $r = p \pm 1$ and prove a proposition that is interesting in its own right.

Proposition 1: For $r = p \pm 1$,

$$\sum_{k=1}^{r-1} \frac{L_k}{F_k} = \Sigma_1 \equiv 0 \pmod{p^2}$$

for any odd prime p.

Proof: In order to show that $\Sigma_1 \equiv 0 \pmod{p^2}$, we need only show that $\Sigma_{k=1}^{r-1}(1/F_kF_{r-k}) \equiv 0 \pmod{p}$ since, from (9), $\Sigma_1 = \Sigma_{k=1}^{r-1}(F_r/F_kF_{r-k})$ and $p|F_r$. In [5], it was proved that $L_{kr} \equiv 2\varepsilon^k \pmod{p^2}$, where ε was as previously defined.

Thus, $L_r \equiv 2\varepsilon \not\equiv 0 \pmod{p}$, and therefore, $\Sigma_{k=1}^{r-1}(1/F_kF_{r-k}) \equiv 0 \pmod{p}$ if and only if $\Sigma_{k=1}^{r-1}(-2L_r/F_kF_{r-k}) \equiv 0 \pmod{p}$. We have, from (8), that

$$\sum_{k=1}^{r-1} \frac{-2L_r}{F_kF_{r-k}} \equiv -5(r-1) + \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 \pmod{p}.$$

We will show that, for $r = p \pm 1$, the right-hand side of the above congruence is congruent to 0 modulo p. We first prove a few simple lemmas.

Lemma 1: The numbers L_k/F_k are all incongruent modulo p for $k = 1, \dots, r-1$.

Proof: Assume that $L_k/F_k \equiv L_j/F_j \pmod{p}$ for some $1 \le j < k \le r-1$. Then $L_kF_j \equiv L_jF_k \pmod{p}$, and from the identity $2F_{k-j} = F_kL_{-j} + F_{-j}L_k$ together with the facts $F_{-j} = (-1)^{j+1}F_j$ and $L_{-j} = (-1)^j L_j$, we obtain $2F_{k-j} = (-1)^j[F_kL_j - F_jL_k] \equiv 0 \pmod{p}$. However, this is impossible because $1 \le k-j \le (r-2)$.

Lemma 2: $(L_k/F_k)^2 \not\equiv 5 \pmod{p}$ for all k and all odd primes p.

Proof: Assume that $(L_k/F_k)^2 \equiv 5 \pmod{p}$, then $L_k^2 \equiv 5F_k^2 \pmod{p}$ so that $2L_k^2 \equiv L_k^2 + 5F_k^2 \pmod{p}$. But, from $2L_{a+b} = L_aL_b + 5F_aF_b$, we have $2L_{2k} = L_k^2 + 5F_k^2$ so that $L_k^2 \equiv L_{2k} \pmod{p}$. However, from the identity $L_{a+b} = L_aL_b - (-1)^b L_{a-b}$, we obtain $L_{2k} = L_k^2 \pm 2$, and combining this with $L_k^2 \equiv L_{2k} \pmod{p}$ we conclude that $0 \equiv \pm 2 \pmod{p}$ for the odd prime p.

We are now in a position to complete the proof of Proposition 1. We have seen that we need to show that $-5(r-1) + \Sigma_{k=1}^{r-1}(L_k/F_k)^2 \equiv 0 \pmod{p}$ for $r = p \pm 1$. We consider the two cases separately.

Case 1. $r = p+1$

$$-5(r-1) + \sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 \equiv -5p + \sum_{k=1}^{p}\left(\frac{L_k}{F_k}\right)^2 \equiv \sum_{k=1}^{p}\left(\frac{L_k}{F_k}\right)^2 \pmod{p}.$$

But from Lemma 1 we have that, for $k = 1, \dots, p = r-1$, the numbers L_k/F_k are all incongruent modulo p; thus, the set of p numbers $\{L_k/F_k : k = 1, \dots, p\}$ forms a complete residue system modulo p. Then

$$\sum_{k=1}^{p}\left(\frac{L_k}{F_k}\right)^2 \equiv \sum_{k=1}^{p} k^2 \equiv 0 \pmod{p}.$$

Case 2. $r = p-1$

$$-5(r-1)+\sum_{k=1}^{r-1}\left(\frac{L_k}{F_k}\right)^2 = -5(p-2)+\sum_{k=1}^{p-2}\left(\frac{L_k}{F_k}\right)^2 \equiv 10+\sum_{k=1}^{p-2}\left(\frac{L_k}{F_k}\right)^2 \pmod{p}.$$

Now all of the L_k / F_k for $k = 1, \dots, p-2$ are incongruent modulo p by Lemma 1 and, from Lemma 2, $(L_k / F_k)^2 \not\equiv 5 \pmod{p}$ for each k. However, 5 is a quadratic residue modulo p [8], and we have

$$10+\sum_{k=1}^{p-2}\left(\frac{L_k}{F_k}\right)^2 \equiv \sum_{k=1}^{p} k^2 \equiv 0 \pmod{p}.$$

Thus, Proposition 1 is proved.

Since $p \nmid L_{mr}$, but $p \mid F_r$ and $F_r \mid F_{mr}$, an immediate consequence of Proposition 1 is the following corollary concerning the last term in equation 10.

Corollary 1:

$$\left(\frac{L_{mr}}{2}\right)^{r-2}\frac{1}{2}\left(F_{mr}+\frac{L_r}{L_{mr}}\frac{F_{mr}^2}{F_r}\right)\Sigma_1 \equiv 0 \pmod{p^3}.$$

Before proving our main theorem, we need the following result about the specific Fibonomial coefficient

$$\begin{bmatrix}(m+1)r-1\\ r-1\end{bmatrix} = \frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k}$$

modulo p^3.

Lemma 3: If $p > 3$ and $r = p \pm 1$, then $\left[{(m+1)r-1 \atop r-1}\right] \equiv (\mp 1)^m \pmod{p^3}$, respectively.

Proof: We again deal with the two cases separately.

Case 1. $r = p-1$

If $r = p-1$, then r is even and $\tau = p-1$. From (10) and Corollary 1,

$$\frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k} \equiv \left(\frac{L_{mr}}{2}\right)^{p-2} - \frac{5}{8}F_{mr}^2\left(\frac{L_{mr}}{2}\right)^{p-4}(p-2) \equiv \frac{1}{4}\left(\frac{L_{mr}}{2}\right)^{p-4}[L_{mr}^2+5F_{mr}^2] \pmod{p^3}.$$

But $L_{mr}^2 + 5F_{mr}^2 = 2L_{2mr}$. Furthermore, $L_{2mr} = L_{mr}L_{mr} - (-1)^{mr}L_{mr-mr} = L_{mr}^2 - 2$, so $L_{mr}^2 + 5F_{mr}^2 = 2(L_{mr}^2 - 2)$. Therefore,

$$\frac{1}{4}\left(\frac{L_{mr}}{2}\right)^{p-4}[L_{mr}^2+5F_{mr}^2] = 2\left(\frac{L_{mr}}{2}\right)^{p-2} - \left(\frac{L_{mr}}{2}\right)^{p-4}.$$

However, from $L_{kr} \equiv 2\varepsilon^k \pmod{p^2}$, we obtain $L_{mr}/2 \equiv 1 \pmod{p^2}$, so $L_{mr}/2 = 1 + p^2 q$ for some q. Then

$$\left(\frac{L_{mr}}{2}\right)^{p-k} = (1+p^2q)^{p-k} \equiv 1+(p-k)p^2q \equiv 1-kp^2q \pmod{p^3},$$

and so

$$2\left(\frac{L_{mr}}{2}\right)^{p-2} - \left(\frac{L_{mr}}{2}\right)^{p-4} \equiv 2(1-2p^2q)-(1-4p^2q) \equiv 1 \pmod{p^3}$$

or

$$\frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k} \equiv 1 \equiv (1)^m \pmod{p^3}.$$

Case 2. $r = p+1$

If $r = p+1$, then $\tau = 2r$ and r is even. From (10) and Corollary 1,

$$\frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k} \equiv \left(\frac{L_{mr}}{2}\right)^p - \frac{5}{8}F_{mr}^2\left(\frac{L_{mr}}{2}\right)^{p-2}(p) \equiv \left(\frac{L_{mr}}{2}\right)^p \pmod{p^3}.$$

Now, $L_{kr} \equiv 2\varepsilon^k \pmod{p^2}$ yields $L_{mr}/2 \equiv (-1)^m \pmod{p^2}$ or $L_{mr}/2 = (-1)^m + p^2q$ for some q. Then,

$$2\left(\frac{L_{mr}}{2}\right)^p \equiv (-1)^{mp} + (-1)^{m(p-1)}(p)(p^2q) \pmod{p^3}$$

or

$$\frac{\prod_{k=1}^{r-1} F_{mr+k}}{\prod_{k=1}^{r-1} F_k} \equiv (-1)^m \pmod{p^3}.$$

Thus, Lemma 3 is proved.

Proposition 2: For any $n \geq 0$ and $m \geq 0$, if $r = p \pm 1$, then

$$\prod_{k=nr+1}^{nr+r-1} F_{mr+k} \equiv (\mp 1)^m \prod_{k=nr+1}^{nr+r-1} F_k \pmod{p^3}, \text{ respectively.}$$

Proof: From Lemma 3,

$$\prod_{k=nr+1}^{nr+r-1} F_{mr+k} = \prod_{k=1}^{r-1} F_{(m+n)r+k} \equiv (\mp 1)^{m+n} \prod_{k=1}^{r-1} F_k \pmod{p^3}$$

and

$$\prod_{k=nr+1}^{nr+r-1} F_k = \prod_{k=1}^{r-1} F_{nr+k} \equiv (\mp 1)^n \prod_{k=1}^{r-1} F_k \pmod{p^3}$$

so that

$$\frac{\prod_{k=nr+1}^{nr+r-1} F_{mr+k}}{\prod_{k=nr+1}^{nr+r-1} F_k} \equiv \frac{(\mp 1)^{m+n}\prod_{k=1}^{r-1} F_k}{(\mp 1)^n \prod_{k=1}^{r-1} F_k} \equiv (\mp 1)^m \pmod{p^3}.$$

Recalling that

$$\begin{bmatrix} a \\ b \end{bmatrix}_r = \frac{F_{ar}F_{(a-1)r}\cdots F_r}{(F_{br}F_{(b-1)r}\cdots F_r)(F_{(a-b)r}\cdots F_r)},$$

we can now prove our main theorem.

Theorem: For any prime $p>3$ and any $a \ge b \ge 0$, if $r = p \pm 1$, then

$$\begin{bmatrix} ra \\ rb \end{bmatrix} \equiv (\mp 1)^{(a-b)b} \begin{bmatrix} a \\ b \end{bmatrix}_r \pmod{p^3}, \text{ respectively.}$$

Proof: Separating the factors divisible by p from those relatively prime to p, we obtain

$$\begin{bmatrix} ar \\ br \end{bmatrix} = \frac{F_{ar}F_{ar-1}\cdots F_{(a-b)r+1}}{F_{br}F_{br-1}\cdots F_1} = \left(\frac{F_{ar}F_{(a-1)r}}{F_{br}F_{(b-1)r}}\cdots\frac{F_{(a-b+1)r}}{F_r}\right)\left(\frac{\prod_{k=(a-1)r+1}^{(a-1)r+r-1} F_k \cdots \prod_{k=(a-b)r+1}^{(a-b)r+r-1} F_k}{\prod_{k=(b-1)r+1}^{(b-1)r+r-1} F_k \cdots \prod_{k=1}^{r-1} F_k}\right).$$

By Proposition 2, the right factor above is congruent to $(\mp 1)^{a-b}\cdots(\mp 1)^{a-b} \equiv (\mp 1)^{(a-b)b} \pmod{p^3}$ and the left factor is $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_r$. Hence,

$$\begin{bmatrix} ar \\ br \end{bmatrix} \equiv (\mp 1)^{(a-b)b} \begin{bmatrix} a \\ b \end{bmatrix}_r \pmod{p^3}.$$

Corollary: For $a \ge b \ge 0$,

$$\begin{bmatrix} a\tau \\ b\tau \end{bmatrix} \equiv \begin{cases} \begin{bmatrix} a \\ b \end{bmatrix}_r & \text{if } r = p-1 \\ \begin{bmatrix} 2a \\ 2b \end{bmatrix}_r & \text{if } r = p+1 \end{cases} \pmod{p^3}.$$

Proof: These follow immediately from the Theorem and the facts: $\tau = p-1$ if $r = p-1$ and $\tau = 2(p+1)$ if $r = p+1$. If $\tau = tr$, then

$$\begin{bmatrix} a\tau \\ b\tau \end{bmatrix} = \begin{bmatrix} atr \\ btr \end{bmatrix} \equiv (\mp 1)^{(a-b)bt} \begin{bmatrix} at \\ bt \end{bmatrix}_r \equiv \begin{bmatrix} ta \\ tb \end{bmatrix}_r \pmod{p^3}.$$

As was shown in [5], if the modulus is only p^2 instead of p^3, the expression $\left[\begin{smallmatrix} ta \\ tb \end{smallmatrix}\right]_r$ can also be written in terms of ordinary binomial coefficients. Can this be done mod p^3 as well? It might also be noted that in [5] this reduction was possible because

$$\frac{F_{kr}}{F_{p^s r}} = \frac{k}{p^s}\left(\frac{L_r}{2}\right)^{k-p^s} \pmod{p^2}$$

if $p^s | k$. (Proposition 2 was the case $s = 0$, but the general case is essentially the same and somewhat more useful.) The same congruence is, in general, false mod p^3.

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