ANTISOCIAL DINNER PARTIES

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1. INTRODUCTION

Take a circular dining table with n chairs arranged at equal intervals around it. In how many ways can ($n/2$ or fewer) people be seated on these chairs in such a way that no two people are seated on adjacent chairs, where two such seating patterns are to be regarded as the same if one can be got from the other by rotation? Let T_n denote the number of such seating patterns (including the case when nobody has come to dinner). The first five values of T_n are $T_1 = 1$, $T_2 = 2$, $T_3 = 2$, $T_4 = 3$, $T_5 = 3$.

I show here that, for $n \geq 2$,

$$T_n = \frac{1}{n} \sum_{d|n} L_d \phi\left(\frac{n}{d}\right) \tag{1}$$

(where the sum is over all positive divisors of n). In this expression, L_d is the d^{th} Lucas number, defined by the recurrence

$$L_0 = 2, L_1 = 1 \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \text{ thereafter,}$$

and ϕ is Euler's totient, i.e.,

$$\phi(n) = \#\{m : 0 \leq m < n, m \text{ and } n \text{ coprime}\}.$$

I was asked this question (i.e., the value of T_n) by my colleague Peter Bushell who, in connection with his work on Shapiro's inequality, was interested in the number of cyclically distinct components of the "regular boundary" of $K = \{\underline{x} \in \mathbf{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. (The regular boundary of K is the set of points $\underline{x}$ in K having at least one component x_i equal to zero, but no two adjacent components both zero and not having x_1 and x_n both zero. See [1] for details and motivation.) So the number he was looking for is $T_n - 1$.

2. THE NUMBER OF CYCLICALLY DISTINCT PATTERNS

Suppose that $n \geq 2$ and let V_n denote the set of all strings $a_1 a_2 \ldots a_n$ on $\{0, 1\}$ of length n such that

V1: a_1 and a_{i+1} are not both 1 (for $1 \leq i < n$),
V2: a_1 and a_n are not both 1.

Such strings correspond to antisocial seating arrangements as described in the introduction (with $a_i = 1$ meaning that chair i is occupied). The cyclic group Z_n acts on V_n by rotations and the number T_n is the number of orbits of this action. Now Burnside's lemma ([2], 10.1.4) tells us that, if G is a finite group acting on a finite set S and if Fix(g) denotes the set of members of S fixed by $g \in C$, then

$$\#\{\text{orbits of G}\} = \frac{1}{\#\text{G}} \sum_{g \in \text{G}} \#\text{Fix}(g).$$

So we need to find #Fix(g) for each element g of Z_n. Suppose that Z_n is generated by σ and that σ acts on V_n by $a_1 a_2 \ldots a_n \sigma = a_n a_1 \ldots a_{n-1}$. It is not difficult to see that, for any integer m, $\text{Fix}(\sigma^m) = \text{Fix}(\sigma^d)$, where $\gcd(m, n) = d$. Now, if d divides n, there are $\phi(n/d)$ integers m with $0 \le m < n$ and $\gcd(m, n) = d$, and it follows that

$$T_n = \#\{\text{orbits of } Z_n\} = \frac{1}{n} \sum_{d|n} \#\text{Fix}(\sigma^d)\phi(n/d).$$

It is plain that

$$\text{Fix}(\sigma^d) = \{\overbrace{vv\ldots v}^{n/d} : v \in V_d\}$$

and so $\#\text{Fix}(\sigma^d) = \#V_d$ and it remains for us to show that

$$\#V_d = L_d. \tag{2}$$

To deal with (2), consider the sets U_n of strings of 0's and 1's that satisfy V1 above but not necessarily V2. Then it is well known (and easily proved) that

$$\#U_n = F_{n+2} \tag{3}$$

(where F_n denotes the n^{th} Fibonacci number) and that

$$L_n = F_{n+1} + F_{n-1}. \tag{4}$$

Let u_n denote a general element of U_n. Now (2) is true when $d = 1$ (V_1 contains only the string 0) and when $d = 2$, so suppose $d \ge 3$ and consider a string v in V_d. Either v ends in 1, in which case it looks like $0u_{d-3}01$, or v ends with 0 and looks like $u_{d-1}0$. By (3), there are F_{d-1} strings of the first kind and F_{d+1} strings of the second kind. Equation (4) now completes the proof of (2) and therefore of (1).

3. A GENERALIZATION

I now consider the obvious extension of the question answered in section 2, that is, what is S_n, the number of seating arrangements that are distinct not only under rotations but also under reflections? Now the group acting on V_n is the dihedral group, D_n, which is generated by σ and the reflection τ, which acts on a string in V_n by writing it backwards (so that $\sigma^n = \tau^2 = 1$ and $\sigma^{-1}\tau = \tau\sigma$).

Let W_k denote the set of palindromic strings w_k in U_k and, for $\eta = 0$ or 1, let $W_{k,\eta}$ denote the subset of W_k of strings that begin (or end) with η. Now $D_n = \{\sigma^m \tau^\varepsilon : 0 \le m < n, \varepsilon = 0, 1\}$, $\text{Fix}(\sigma^m)$ is as in section 1, and (as is easy to see) $\text{Fix}(\sigma^m \tau)$ is made up of strings $w_{n-m} w_m$, where w_{n-m} and w_m do not both have 1's at each end. So

$$\#\text{Fix}(\sigma^m \tau) = \#W_{n-m,0}\#W_{m,0} + \#W_{n-m,1}\#W_{m,0} + \#W_{n-m,0}\#W_{m,1}. \tag{5}$$

Using (3), we have, for even k,

$$\#W_{k,1} = \#U_{k/2-3} = F_{k/2-1},$$
$$\#W_{k,0} = \#U_{k/2-2} = F_{k/2},$$

and, if k is odd,

$$\begin{aligned} \#W_{k,1} &= \#U_{(k-5)/2} + \#U_{(k-7)/2} = F_{(k+1)/2}, \\ \#W_{k,0} &= \#U_{(k-3)/2} + \#U_{(k-5)/2} = F_{(k+3)/2}. \end{aligned} \tag{6}$$

With the help of the well-known identity, $F_{r-1}F_s + F_rF_{s+1} = F_{r+s}$, it follows from (5) and (6) that

$$\#\mathrm{Fix}(\sigma^m\tau) = \begin{cases} F_{n/2} & \text{if } n \text{ and } m \text{ are even,} \\ F_{n/2+3} & \text{if } n \text{ is even and } m \text{ is odd,} \\ F_{(n+3)/2} & \text{if } n \text{ is odd,} \end{cases}$$

and Burnside's lemma gives

$$S_n = \begin{cases} \dfrac{1}{2n}\sum_{d|n} L_d\phi(n/d) + \dfrac{1}{2}F_{n/2+2} & \text{if } n \text{ is even,} \\ \dfrac{1}{2n}\sum_{d|n} L_d\phi(n/d) + \dfrac{1}{2}F_{(n+3)/2} & \text{if } n \text{ is odd.} \end{cases}$$

REFERENCES

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2. W. R. Scott. *Group Theory.* Englewood Cliffs, N.J.: Prentice Hall, 1964.

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