A NOTE REGARDING CONTINUED FRACTIONS

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In this note we develop some properties of purely periodic infinite continued fractions. The parameters k, n, a_k, a_n, p_n, and q_n will denote positive integers, and $q_0 = 0$. Let

$$y_n = [a_1, a_2, a_3, \ldots, a_n] = p_n / q_n,$$

that is, y_n is the finite continued fraction whose partial quotients are the a_k. (The initial term of y_n is denoted a_1, not a_0.) Let

$$x_n = [\overline{a_1, a_2, a_3, \ldots, a_n}],$$

that is, x_n is the corresponding purely periodic infinite continued fraction.

Theorem 1: Let n, x_n, y_n, p_n, and q_n be as above. Then

$$x_n = \left(p_n - q_{n-1} + \sqrt{(p_n + q_{n-1})^2 - 4(-1)^n}\right) / 2q_n.$$

Proof: This follows from elementary considerations (see Hardy & Wright [1], Ch. 10). □

Remark: S. Rabinowitz [3] has asked for a formula for $[\overline{1, 2, 3, \ldots, n}]$.

Theorem 2: Let n, x_n, y_n, p_n, and q_n be as above. Let

$$\lim_{n\to\infty} y_n = A = [a_1, a_2, a_3, \ldots].$$

Then also

$$\lim_{n\to\infty} x_n = A.$$

Proof: It suffices to show that $y_n - x_n$ tends to 0 as n tends to infinity. By Theorem 1, we have

$$y_n - x_n = \frac{1}{2q_n(p_n + q_{n-1})}\left(1 - \left(1 - \frac{4(-1)^n}{(p_n + q_{n-1})^2}\right)^{1/2}\right).$$

As n tends to infinity, the factor $\frac{1}{2q_n}(p_n + q_{n-1})$ is bounded from above, since p_n / q_n tends to A and $q_{n-1} / q_n < 1$. On the other hand, p_n and q_{n-1} tend to infinity with n, so that

$$1 - \left(1 - \frac{4(-1)^n}{(p_n + q_{n-1})^2}\right)^{1/2} \quad \text{tends to } 0.$$

Thus, $y_n - x_n$ tends to 0 as n tends to infinity. □

Corollary: Let $I_k(t)$ be the modified Bessel function of the first kind of order k, that is,

$$I_k(t) = \sum_{j=0}^{\infty} (\tfrac{1}{2}t)^{2j+k} / \Gamma(j+1)\Gamma(j+k+1).$$

Let $w_n = [\overline{1, 2, 3, \ldots, n}]$. Then

$$\lim_{n\to\infty} w_n = I_0(2) / I_1(2) = 1.433127427.$$

Proof: This follows from hypothesis, Theorem 2, and ([2], Th. 1). □

Theorem 3: Let x_n, y_n, and A be as in the hypothesis of Theorem 2. Then, for all n, we have $x_{2n} < A < x_{2n-1}$.

Proof: Applying Theorem 1, we have $x_{2n} < p_{2n} / q_{2n}$, that is, $x_{2n} < y_{2n}$. Similarly, $x_{2n-1} > p_{2n-1} / q_{2n-1}$, that is, $x_{2n-1} > y_{2n-1}$. But $y_{2n} < A < y_{2n-1}$ for all n, so $x_{2n} < A < x_{2n-1}$ for all n. □

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 6th ed. Oxford: Oxford University Press, 1972.
2. D. H. Lehmer. "Continued Fractions Containing Arithmetic Progressions." *Scripta Math.* **39** (1971):17-23.
3. S. Rabinowitz. Problem posed at the Fourth International Conference on Fibonacci Numbers and Their Applications, Wake Forest University, 1990.

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