

A NOTE REGARDING CONTINUED FRACTIONS

Neville Robbins

Mathematics Department, San Francisco State University, San Francisco, CA 94132

(Submitted November 1993)

In this note we develop some properties of purely periodic infinite continued fractions. The parameters $k, n, a_k, a_n, p_n,$ and q_n will denote positive integers, and $q_0 = 0$. Let

$$y_n = [a_1, a_2, a_3, \dots, a_n] = p_n / q_n,$$

that is, y_n is the finite continued fraction whose partial quotients are the a_k . (The initial term of y_n is denoted a_1 , not a_0 .) Let

$$x_n = [\overline{a_1, a_2, a_3, \dots, a_n}],$$

that is, x_n is the corresponding purely periodic infinite continued fraction.

Theorem 1: Let $n, x_n, y_n, p_n,$ and q_n be as above. Then

$$x_n = \left(p_n - q_{n-1} + \sqrt{(p_n + q_{n-1})^2 - 4(-1)^n} \right) / 2q_n.$$

Proof: This follows from elementary considerations (see Hardy & Wright [1], Ch. 10). \square

Remark: S. Rabinowitz [3] has asked for a formula for $[1, 2, 3, \dots, n]$.

Theorem 2: Let $n, x_n, y_n, p_n,$ and q_n be as above. Let

$$\lim_{n \rightarrow \infty} y_n = A = [a_1, a_2, a_3, \dots].$$

Then also

$$\lim_{n \rightarrow \infty} x_n = A.$$

Proof: It suffices to show that $y_n - x_n$ tends to 0 as n tends to infinity. By Theorem 1, we have

$$y_n - x_n = \frac{1}{2q_n(p_n + q_{n-1})} \left(1 - \left(1 - \frac{4(-1)^n}{(p_n + q_{n-1})^2} \right)^{1/2} \right).$$

As n tends to infinity, the factor $\frac{1}{2q_n}(p_n + q_{n-1})$ is bounded from above, since p_n / q_n tends to A and $q_{n-1} / q_n < 1$. On the other hand, p_n and q_{n-1} tend to infinity with n , so that

$$1 - \left(1 - \frac{4(-1)^n}{(p_n + q_{n-1})^2} \right)^{1/2} \text{ tends to } 0.$$

Thus, $y_n - x_n$ tends to 0 as n tends to infinity. \square

Corollary: Let $I_k(t)$ be the modified Bessel function of the first kind of order k , that is,

$$I_k(t) = \sum_{j=0}^{\infty} (\frac{1}{2}t)^{2j+k} / \Gamma(j+1)\Gamma(j+k+1).$$

Let $w_n = \overline{[1, 2, 3, \dots, n]}$. Then

$$\lim_{n \rightarrow \infty} w_n = I_0(2) / I_1(2) = 1.433127427.$$

Proof: This follows from hypothesis, Theorem 2, and ([2], Th. 1). \square

Theorem 3: Let $x_n, y_n,$ and A be as in the hypothesis of Theorem 2. Then, for all n , we have $x_{2n} < A < x_{2n-1}$.

Proof: Applying Theorem 1, we have $x_{2n} < p_{2n} / q_{2n}$, that is, $x_{2n} < y_{2n}$. Similarly, $x_{2n-1} > p_{2n-1} / q_{2n-1}$, that is, $x_{2n-1} > y_{2n-1}$. But $y_{2n} < A < y_{2n-1}$ for all n , so $x_{2n} < A < x_{2n-1}$ for all n . \square

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 6th ed. Oxford: Oxford University Press, 1972.
2. D. H. Lehmer. "Continued Fractions Containing Arithmetic Progressions." *Scripta Math.* **39** (1971):17-23.
3. S. Rabinowitz. Problem posed at the Fourth International Conference on Fibonacci Numbers and Their Applications, Wake Forest University, 1990.



GENERALIZED PASCAL TRIANGLES AND PYRAMIDS THEIR FRACTALS, GRAPHS, AND APPLICATIONS

by Dr. Boris A. Bondarenko
Associate member of the Academy of Sciences of the Republic of Uzbekistan, Tashkent

Translated by Professor Richard C. Bollinger
Penn State at Erie, The Behrend College

This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book see *The Fibonacci Quarterly*, Volume 31.1, page 52.

The translation of the book is being reproduced and sold with the permission of the author, the translator and the "FAN" Edition of the Academy of Science of the Republic of Uzbekistan. The book, which contains approximately 250 pages, is a paper back with a plastic spiral binding. The price of the book is \$31.00 plus postage and handling where postage and handling will be \$6.00 if mailed to anywhere in the United States or Canada, \$9.00 by surface mail or \$16.00 by airmail to anywhere else. A copy of the book can be purchased by sending a check made out to **THE FIBONACCI ASSOCIATION** for the appropriate amount along with a letter requesting a copy of the book to: **RICHARD VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.**