AN EXTENSION OF STIRLING NUMBERS

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(Submitted August 1994)

1. INTRODUCTION

Stirling numbers may be defined as the coefficients in an expansion of positive integral powers of a variable in terms of factorial powers, or vice-versa:

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k, \quad n \geq 0, \tag{1.1}$$

$$x^n = \sum_{k=0}^{n} S(n, k)(x)_k, \quad n \geq 0, \tag{1.2}$$

where

$$\begin{aligned} &(x)_n = x(x-1)\cdots(x-n+1), \quad n \geq 1, \\ &(x)_0 = 1. \end{aligned} \tag{1.3}$$

The numbers $s(n, k)$ and $S(n, k)$ are, in the notation of Riordan [6], Stirling numbers of the first and second kind, respectively.

Writing (1.3) as

$$(x)_n = \Gamma(x+1) / \Gamma(x-n+1) \tag{1.4}$$

we may generalize factorial powers to negative integral values of n:

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)}, \quad n \geq 1.$$

The question then arises as to whether we can extend Stirling numbers to negative integral values of one or both of their arguments. Several authors have discussed the case where both n and k are negative integers (for a brief history, see Knuth [4]), and we briefly discuss this case in Section 3. We shall refer to such numbers as Negative-Negative Stirling Numbers (NNSN) whereas we call the numbers defined by (1.1) and (1.2) Positive-Positive Stirling Numbers (PPSN). It is clearly impossible to have an expansion of the form (1.1) or (1.2), where n is a positive integer and k is summed over negative values: in the first case, the left-hand side is a polynomial whereas the right-hand side has a singularity at $x = 0$; in the second case, taking the limit as x goes to infinity, the left-hand side goes to infinity whereas the right-hand side goes to zero (for a proof of the uniform convergence required to take the limit term-by-term, see, for example, Milne-Thomson [5]). However, it is possible to extend (1.1) and (1.2) to the case where n is a negative integer and k is summed over positive integers. We call the resulting coefficients Negative-Positive Stirling Numbers (NPSN), and the purpose of this article is to discuss these numbers and some of their properties.

In Section 2 we summarize some well-known properties of PPSN. In Section 3 we describe four urn models: the first two illustrate PPSN and the second two demonstrate the connection between NNSN and PPSN. We define NPSN in Section 4 and obtain explicit expressions for them by means of two further urn models. Finally, in Section 5 we give some alternative representations of NPSN, tabulate some values and derive some properties.

2. PROPERTIES OF POSITIVE-POSITIVE STIRLING NUMBERS

In this section we list, for future reference, some well-known properties of PPSN. For more details, see, for example, Jordan [3], Chapter IV.

By using the definition (1.1) and the identity

$$(x)_{n+1} = (x)_n(x-n), \tag{2.1}$$

and by equating coefficients of powers of x, we may derive the recurrence relation

$$s(n+1, k) = s(n, k-1) - ns(n, k). \tag{2.2}$$

Similarly, the definition (1.2) and the identity

$$x^{n+1} = x^n(x-k+k), \tag{2.3}$$

and the equating of factorial powers of x, leads to

$$S(n+1, k) = S(n, k-1) + kS(n, k). \tag{2.4}$$

Numerical values for PPSN may readily be generated using (2.2) or (2.4) together with the boundary values [which follow immediately from (1.1) and (1.2)]

$$s(n, n) = 1, \quad s(n+1, 0) = 0, \quad n \geq 0, \tag{2.5}$$

$$S(n, n) = 1, \quad S(n+1, 0) = 0, \quad n \geq 0. \tag{2.6}$$

We may define a generating function for $S(n, k)$ (with respect to n) by

$$A_k(t) = \sum_{n=k}^{\infty} t^n S(n, k).$$

By using (2.4), we can obtain a first-order linear difference equation for A_k which, with the initial condition $A_0(t) = 1$, can be solved to give

$$\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} = \sum_{n=k}^{\infty} t^n S(n, k), \quad k \geq 1. \tag{2.7}$$

The left-hand side of (2.7) can be expressed in partial fractions:

$$\frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\frac{1}{1-rt}.$$

Expanding the last factor by the binomial theorem and comparing with (2.7) gives the following representation, known as Stirling's formula:

$$S(n, k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n. \tag{2.8}$$

3. URN MODELS AND STIRLING NUMBERS

In all the models of this section and the next, an urn contains white balls and black balls. Some, but not necessarily all, of the balls are of unit mass. Balls are drawn one at a time with replacement, and the probability that a particular ball is drawn is proportional to its mass.

Model A

An urn originally contains only white balls of total mass x $(> n-1)$ including at least $n-1$ balls of unit mass. We successively draw n balls with replacement. After a white ball is drawn (and replaced) and before the next ball is drawn, we substitute a unit mass white ball in the urn by a unit mass black ball. Then the probability that the number of white balls drawn is k is equal to

$$P_n^A(k) = \Sigma \frac{x}{x}\left(\frac{1}{x}\right)^{a_1} \frac{x-1}{x}\left(\frac{2}{x}\right)^{a_2} \frac{x-2}{x} \cdots \left(\frac{k-1}{x}\right)^{a_{k-1}} \frac{x-k+1}{x}\left(\frac{k}{x}\right)^{a_k}, \quad a \le k \le n, \tag{3.1}$$

where the sum is over all nonnegative integers a_i satisfying $a_1 + \cdots + a_k = n-k$. It is clear that, for $n \ge 1$, $P_n^A(0) = 0$, so (3.1) may be rewritten as

$$P_n^A(k) = \frac{(x)_k}{x^n} b(n, k), \tag{3.2}$$

with

$$b(n, k) = \Sigma u_1 u_2 \cdots u_{n-k}, \quad 1 \le k < n, \tag{3.3}$$

where the sum is over all integers u_i $(i = 1, 2, \ldots, n-k)$ satisfying

$$1 \le u_1 \le u_2 \le \cdots \le u_{n-k} \le k;$$
$$b(n, 0) = 0, \qquad b(n, n) = 1.$$

In terms of the urn model, if $n = 0$, then certainly $k = 0$, so (3.2) continues to hold if we put $b(0, 0) = 1$.

From the condition $\Sigma_{k=0}^{n} P_n^A(k) = 1$ and equation (3.2), we deduce

$$x^n = \sum_{k=0}^{n} (x)_k b(n, k).$$

Comparing this expression with (1.2), we conclude that a representation for $S(n, k)$ is provided by

$$S(n, k) = b(n, k). \tag{3.4}$$

Model B

An urn originally contains only white balls of total mass y. We successively draw n balls with replacement. After each ball is drawn (and replaced) and before the next ball is drawn, we add one black ball of unit mass to the urn. The probability that the number of white balls drawn is k is equal to

$$P_n^B(k) = \frac{y^k \Sigma v_1 v_2 \cdots v_{n-k}}{y(y+1)\cdots(y+n-1)}, \quad 1 \le k < n, \tag{3.5}$$

where the sum is over all integers v_i $(i = 1, 2, \ldots, n-k)$ satisfying

$$1 \le v_1 < v_2 < \cdots < v_{n-k} \le n-1. \tag{3.6}$$

The factors in the denominator of (3.5) represent the total mass of the balls in the urn at successive drawings; the factor y^k arises from drawing a white ball on k occasions, and the factor v_i arises from the drawing of a black ball at a time when the urn contains v_i black balls.

Note that, for $n \geq 1$, $P_n^B(0) = 0$ and $P_n^B(n) = y^n / [y(y+1) \cdots (y+n-1)]$, so that (3.5) may be written as

$$P_n^B(k) = \frac{y^k c(n,k)}{y(y+1) \cdots (y+n-1)}, \quad 0 \leq k \leq n, \tag{3.7}$$

with

$$c(n,k) = \sum v_1 v_2 \cdots v_{n-k}, \quad 1 \leq k < n, \tag{3.8}$$

under the restriction (3.6); $c(n,0) = 0$, $c(n,n) = 1$.

Putting $y = -x$, (3.7) may be written as

$$P_n^B(k) = \frac{x^k}{(x)_n} (-1)^{n+k} c(n,k). \tag{3.9}$$

As in Model A, if $n = 0$ then $k = 0$, so (3.9) holds if $c(0,0) = 1$. The probabilities in (3.9) must add to one; hence,

$$(x)_n = \sum_{k=0}^{n} x^k (-1)^{n+k} c(n,k).$$

Comparing this with (1.1), we see that

$$s(n,k) = (-1)^{n+k} c(n,k) \tag{3.10}$$

where $c(n,k)$ is given by (3.8).

The PPSN representations derived above by means of Model A and Model B are, of course, well known and have been derived by other methods (see, for example, Jordan [3]). We now consider two urn models of relevance to Negative-Negative Stirling Numbers. By analogy with (1.1) and (1.2) the NNSN of the first and second kind, respectively, are defined by

$$(x)_{-n} = \sum_{k=0}^{\infty} s(-n,-k) x^{-k}, \quad n \geq 0, \tag{3.11}$$

$$x^{-n} = \sum_{k=0}^{\infty} S(-n,-k)(x)_{-k}, \quad n \geq 0. \tag{3.12}$$

Model C

The situation here is the same as in Model B except that now we continue to draw balls until we have drawn $n+1$ white balls. The probability that the total number of balls drawn is $k+1$ is equal to

$$P_{n+1}^C(k+1) = \frac{y^{n+1} \sum v_1 v_2 \cdots v_{k-n}}{y(y+1) \cdots (y+k)}, \quad 1 \leq n < k, \tag{3.13}$$

where $1 \leq v_1 < v_2 < \cdots < v_{k-n} \leq k-1$. If all the balls drawn are white, we have

$$P_{n+1}^C(n+1) = y^{n+1} / [y(y+1) \cdots (y+n)]$$

whereas, if $n = 0$, then certainly $k = 0$. It follows from (3.8) that (3.13) may be written as

$$P_{n+1}^C(k+1) = y^n (y)_{-k} c(k,n), \quad 0 \leq n \leq k. \tag{3.14}$$

Suppose that after the $(i+1)^{\text{th}}$ ball is drawn all the remaining balls drawn are black. The probability of this event is given by the infinite product

$$\prod_{m=1}^{\infty}\frac{i+m}{y+i+m}=\prod_{m=1}^{\infty}\left(1-\frac{y}{y+i+m}\right)=0$$

(see, for example, Ferrar [2], p. 147). It follows that the probability distribution (3.14) is proper and sums to one. Hence,

$$y^{-n}=\sum_{k=n}^{\infty}(y)_{-k}c(k,n).$$

The coefficients in an expansion such as this are unique (see Milne-Thomson [5], p. 288), so, by comparison with (3.12) and using (3.10), we see that

$$S(-n,-k)=c(k,n)=(-1)^{n+k}s(k,n),\quad 0\le n\le k \tag{3.15}$$

[and $S(-n,-k)=0$ for $0\le k<n$].

Model D

This is the same as Model A, except that the white balls originally in the urn have total mass x ($>n$) and include at least n balls of unit mass. We now continue to draw balls until we have drawn $n+1$ white balls. The probability that the total number of balls drawn is $k+1$ is equal to

$$P_{n+1}^{D}(k+1)=\Sigma\frac{x}{x}\left(\frac{1}{x}\right)^{a_1}\frac{x-1}{x}\left(\frac{2}{x}\right)^{a_2}\frac{x-2}{x}\cdots\left(\frac{n}{x}\right)^{a_n}\frac{x-n}{x},\quad 0\le n\le k,$$

where $a_1+\cdots+a_n=k-n$.

Using the notation of (3.3), we can write

$$P_{n+1}^{D}(k+1)=\frac{x(x-1)\cdots(x-n)b(k,n)}{x^{k+1}},\quad 0\le n\le k. \tag{3.16}$$

The probability that after the i^{th} white ball is drawn all the remaining balls drawn are black is

$$\lim_{m\to\infty}\left(\frac{i}{x}\right)^{m}=0,$$

so the probability distribution (3.16) is proper and sums to one.

Putting $x=-y$, (3.16) may be written as

$$P_{n+1}^{D}(k+1)=\frac{(-1)^{n+k}}{(y)_{-n}y^{k}}b(k,n),\quad 0\le n\le k.$$

If we sum over k and rearrange this equation, we obtain

$$(y)_{-n}=\sum_{k=n}^{\infty}y^{-k}(-1)^{n+k}b(k,n).$$

Comparing this with (3.11) (using the uniqueness of Laurent series coefficients) and using (3.4), we conclude

$$s(-n,-k)=(-1)^{n+k}b(k,n)=(-1)^{n+k}S(k,n),\quad 0\le n\le k, \tag{3.17}$$

[and $s(-n,-k)=0$ for $0\le k<n$].

Equations (3.15) and (3.17) show the interesting fact, noted by other authors (see Knuth [4]), that, apart from a sign, NNSN of the first (second) kind are obtained from PPSN of the second (first) kind by a reflection in the line $n = -k$.

4. NEGATIVE-POSITIVE STIRLING NUMBERS

Negative-Positive Stirling Numbers of the first and second kind may be defined by the obvious modifications of (1.1) and (1.2), respectively:

$$(x)_{-n} = \sum_{k=0}^{\infty} s(-n, k)x^k, \quad n \geq 1, \tag{4.1}$$

$$x^{-n} = \sum_{k=1}^{\infty} S(-n, k)(x)_k, \quad n \geq 0. \tag{4.2}$$

(The reason for the limits on n and k will become clear later.)

Two further urn models allow us to give explicit representations for these NPSN.

Model E

Originally the urn contains white balls of mass x (< 1) and black balls of mass $1-x$. Balls are drawn one at a time with replacement. After a black ball is drawn (and replaced) and before the next ball is drawn, we add one black ball of unit mass to the urn. We continue until $n+1$ white balls have been drawn. For $n \geq 0$, the probability that the number of black balls drawn is $k-1$ is equal to

$$P_{n+1}^E(k-1) = \Sigma\left(\frac{x}{1}\right)^{a_1} \frac{1-x}{1}\left(\frac{x}{2}\right)^{a_2} \frac{2-x}{2} \cdots \left(\frac{x}{k-1}\right)^{a_{k-1}} \frac{k-1-x}{k-1}\left(\frac{x}{k}\right)^{a_k+1}, \quad k \geq 2,$$

where the sum is over all nonnegative integers a_i satisfying $a_1 + \cdots + a_k = n$; if all the balls drawn are white, then $P_{n+1}^E(0) = x^{n+1}$. We may therefore write

$$P_{n+1}^E(k-1) = x^n (x)_k (-1)^{k-1} a(n, k) / k! \tag{4.3}$$

with

$$a(n, k) = \Sigma \frac{1}{u_1 u_2 \cdots u_n}, \quad n \geq 1, \tag{4.4}$$

where the sum is over all integers u_i $(i = 1, 2, \ldots, n)$ satisfying

$$1 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq k;$$
$$a(0, k) = 1.$$

The probability that after the i^{th} black ball is drawn all the remaining balls drawn are black is

$$\prod_{m=1}^{\infty} \frac{i+m-x}{i+m} = \prod_{m=1}^{\infty}\left(1 - \frac{x}{i+m}\right) = 0.$$

We conclude that the probabilities in (4.3) sum to one, and hence,

$$x^{-n} = \sum_{k=1}^{\infty} (x)_k (-1)^{k-1} a(n, k) / k!. \tag{4.5}$$

An expansion of x^{-n} of this form in terms of factorial powers $(x)_k$ has unique coefficients. The reason for this is that we have effectively put the coefficient of $(x)_0$ equal to zero (see Milne-Thomsom [5], pp. 305-06).

Comparison of (4.5) and (4.2) gives

$$S(-n, k) = (-1)^{k-1}a(n, k) / k!, \tag{4.6}$$

with $a(n, k)$ given by (4.4).

Model F

The rules for this model are the same as for Model E, except that now we continue until n *black* balls have been drawn. For $n \geq 1$, the probability that the number of white balls drawn is k is equal to

$$P_n^F(k) = \Sigma\left(\frac{x}{1}\right)^{a_1}\frac{1-x}{1}\left(\frac{x}{2}\right)^{a_2}\frac{2-x}{2}\cdots\left(\frac{x}{n}\right)^{a_n}\frac{n-x}{n}, \quad k \geq 0,$$

where $a_1 + \cdots + a_n = k$.

Using (4.4), we may write

$$P_n^F(k) = x^k(1-x)\cdots(n-x)a(k, n) / n!. \tag{4.7}$$

The probability that after the $(i-1)^{\text{th}}$ black ball is drawn all the remaining balls drawn are white is

$$\lim_{m\to\infty}\left(\frac{x}{i}\right)^m = 0$$

so, again, the probabilities must add to one.

Putting $x = -y$, (4.7) becomes

$$P_n^F(k) = \frac{y^k}{(y)_{-n}}(-1)^k a(k, n) / n!.$$

Summing over k gives

$$(y)_{-n} = \sum_{k=0}^{\infty} y^k(-1)^k a(k, n) / n!,$$

and a comparison with (4.1) gives

$$s(-n, k) = (-1)^k a(k, n) / n!. \tag{4.8}$$

5. PROPERTIES OF NEGATIVE-POSITIVE STIRLING NUMBERS

At the end of Section 3 we noted that

$$|S(-n, -k)| = |s(k, n)| \tag{5.1}$$

whenever n and k have the same sign. Equations (4.6) and (4.8) show that NPSN are related by the same reflection in the line $n = -k$, that is, (5.1) continues to hold if n is positive and k is negative. The consequence is that NPSN of the first and second kinds are (apart from a sign) the same set of numbers, but differently indexed, so that any property of NPSN of one kind can be immediately expressed as a property of the other kind. Explicitly,

$$S(-n, k) = (-1)^{n+k-1} s(-k, n). \tag{5.2}$$

Different Representations

If we regard (4.1) as a Taylor series, it follows, using (5.2), that

$$S(-n, k) = \frac{(-1)^{n+k-1}}{n!}\left[\frac{d^n}{dx^n}\left(\frac{1}{(x+1)(x+2)\cdots(x+k)}\right)\right]_{x=0}. \tag{5.3}$$

It is not difficult to show directly the equivalence of (4.6) and (5.3).

If we expand (5.3) by partial fractions, perform the n-fold differentiation, and put $x = 0$, we obtain

$$S(-n, k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^{-n}. \tag{5.4}$$

Thus, we note that Stirling's formula (2.8) continues to hold if n is negative.

Recurrence Formulas and Table of Values

The recurrence relations for PPSN, (2.2) and (2.4), were derived from the identities (2.1) and (2.3). These identities hold whatever the sign of n, and it follows that (2.2) and (2.4) continue to hold for NPSN. These relations can therefore be used, together with appropriate boundary values, to generate numerical values. For NPSN of the first kind, we may rewrite (2.2) as

$$s(-n+1, k) = s(-n, k-1) + ns(-n, k), \quad n \geq 2,\ k \geq 1. \tag{5.5}$$

On putting, respectively, $n = 1$ and $x = 0$ in the definition (4.1), we obtain

$$s(-1, k) = (-1)^k, \qquad s(-n, 0) = 1/n!. \tag{5.6}$$

Combining (5.5) with (5.6), we can generate the values for $s(-n, k)$ given in Table 1. Values for $S(-n, k)$ are then given by (5.2).

TABLE 1. Values of $s(-n, k)$

$n \backslash k$	0	1	2	3
1	1	-1	1	-1
2	$\frac{1}{2}$	$-\frac{3}{4}$	$\frac{7}{8}$	$-\frac{15}{16}$
3	$\frac{1}{6}$	$-\frac{11}{36}$	$\frac{85}{216}$	$-\frac{575}{1296}$
4	$\frac{1}{24}$	$-\frac{25}{288}$	$\frac{415}{3456}$	$-\frac{5845}{41472}$

Generating Functions

We derive a number of generating functions for NPSN. They can easily be translated into generating functions for the other kind of NPSN by use of (5.2).

The definition (4.1) itself provides a generating function for $s(-n, k)$. An exponential generating function can be obtained from (5.4):

$$\sum_{n=0}^{\infty} S(-n, k)\frac{y^n}{n!} = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}\exp\left(\frac{y}{r}\right).$$

Two double generating functions can be obtained from (4.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} s(-n, k)x^k y^n = \sum_{n=1}^{\infty}\frac{\Gamma(x+1)}{\Gamma(x+n+1)}y^n \qquad \text{by (4.1) and (1.4)}$$

$$= M(1, 1+x, y) - 1,$$

where M is a confluent hypergeometric function (see [1], p. 504). Similarly,

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} s(-n, k)x^k \frac{y^n}{n!} = {}_0F_1(1+x, y) - 1$$

$$= [\Gamma(x+1)I_x(2\sqrt{y})/y^{x/2}] - 1,$$

where ${}_0F_1$ and I_x are, respectively, a generalized hypergeometric function and a modified Bessel function (see [1], pp. 556, 374, 377).

Another pair of double generating functions can be obtained from (4.2):

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} S(-n, k)(x)_k y^n = \frac{x}{x-y},$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} S(-n, k)(x)_k \frac{y^n}{n!} = \exp(y/x).$$

Asymptotic Behavior

If n is taken to infinity in (5.4), only the $r = 1$ term survives. Hence,

$$\lim_{n\to\infty} S(-n, k) = (-1)^{k-1}/(k-1)!.$$

The definition (4.4) implies that we can express $a(n, k)$ as $h_n(1, \frac{1}{2}, \ldots, \frac{1}{k})$, where h_n is a homogeneous product sum symmetric function (see Riordan [6], p. 47). Riordan shows that $n!h_n$ can be expressed as a (Bell) polynomial Y_n in the variables s_i $(i = 1, \ldots, n)$, where

$$s_i = 1 + \frac{1}{2^i} + \cdots + \frac{1}{k^i}.$$

As $k \to \infty$, all s_i tend to a finite limit apart from s_1 which behaves like $\ln k$. It is clear that the term involving the highest power of s_1 in Y_n is s_1^n. Hence, as $k \to \infty$, $Y_n \sim (\ln k)^n$. From (4.6), we conclude that

$$|S(-n, k)| \sim \frac{(\ln k)^n}{k!n!} \text{ as } k \to \infty.$$

Orthogonality and Other Relations

For $n \geq m \geq 1$, we have

$$(x)_n/(x)_m = (x-m)(x-m-1)\cdots(x-n+1) = (x-m)_{n-m},$$

and

$$\frac{1}{(x)_m} = \frac{1}{x(x-1)\cdots(x-m+1)} = \frac{(-1)^{m-1}}{x(-x+1)\cdots(-x+m-1)} = \frac{(-1)^{m-1}(-x)_{-m+1}}{x}.$$

Hence,

$$(x-m)_{n-m} = (-1)^{m-1}(x)_n(-x)_{-m+1}/x. \tag{5.7}$$

If, for $m \geq 2$, we expand the factorial powers by (1.1) and (4.1) we obtain

$$\sum_{k=0}^{n-m} s(n-m,k)(x-m)^k = (-1)^{m-1}x^{-1}\sum_{p=0}^{n} s(n,p)x^p \sum_{q=0}^{\infty} s(-m+1,q)(-x)^q. \tag{5.8}$$

Expanding the term $(x-m)^k$ on the left-hand side by the binomial theorem, and equating coefficients of like powers of x, gives, for $0 \leq r \leq n-m$, the following relation between PPSN and NPSN

$$\sum_{k=r}^{n-m}(-1)^k\binom{k}{r}s(n-m,k)m^{k-r} = \sum_{p=0}^{r+1}(-1)^{m+p}s(n,p)s(-m+1,r+1-p), \tag{5.9}$$

and for $r > n-m$, the orthogonality relation

$$\sum_{p=0}^{\min(r+1,n)}(-1)^p s(n,p)s(-m+1,r+1-p) = 0. \tag{5.10}$$

When $m > n \geq 1$, the left-hand side of (5.7) is replaced by

$$\frac{1}{(x-n)(x-n-1)\cdots(x-m+1)}. \tag{5.11}$$

If we express this function in partial fractions and then expand each term as a power series in x, we can again equate the coefficients of powers of x with those on the right-hand side of (5.8), obtaining, for $r \geq 0$,

$$\sum_{k=n}^{m-1}\frac{(-1)^k}{(k-n)!(m-1-k)!k^{r+1}} = \sum_{p=0}^{\min(r+1,n)}(-1)^{p+r}s(n,p)s(-m+1,r+1-p). \tag{5.12}$$

It is possible to obtain equivalent results involving NNSN [and hence, by (3.17), PPSN] instead of NPSN by using (3.11) to expand the term $(-x)_{-m+1}$ in (5.7). For $n \geq m \geq 1$ and $0 \leq r \leq n-m$, the only difference from (5.9) is that the sum on the right-hand side now goes from $p = m+r$ to $p = n$; if $r < 0$, the sum on the left-hand side of (5.10) now goes from $p = \max(m+r, 0)$ to $p = n$. Similarly, if $m > n > 0$, we expand the terms in the partial fraction version of (5.11) as power series in $(1/x)$ and equate coefficients of like powers. For $r \leq -(m-n)$, the right-hand side of (5.12) acquires an extra factor of (-1) and the sum now goes from $p = \max(m+r, 0)$ to n.

ACKNOWLEDGMENT

I should like to thank the referee for some helpful suggestions.

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AMS Classification Numbers: 11B73, 60C05

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