

GENERALIZED HOCKEY STICK IDENTITIES AND N-DIMENSIONAL BLOCKWALKING

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1. INTRODUCTION

Traditionally, the word "abracadabra" was encrypted onto amulets and other magical paraphernalia to help ward off evil. George Pólya ([8], [9]) provided the cryptic form of this word shown in Figure 1 and asked how many ways abracadabra can be spelled out using this diagram. If we replace the diagram in Figure 1 with the grid shown in Figure 2 where letters in the original diagram are placed at points of intersection of the grid, then the question is equivalent to asking how many paths there are from the top of the grid to the spade at the bottom of the grid where each step is either down and left or down and right. One such path that spells out abracadabra is shown. By considering this grid as a city street map, we can think of such paths as walking city blocks—or *blockwalking*.

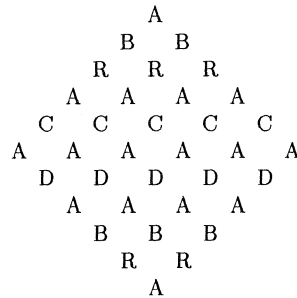


FIGURE 1. How many ways can you spell ABRACADABRA in the above diagram?

We can represent a blockwalking path by a series of *R*'s (right turns) and *L*'s (left turns). For example, the abracadabra path illustrated in Figure 2 can be represented by *LRLRLRLRLR*. It is easy to see that any path which spells abracadabra will have 5 *R*'s and 5 *L*'s. More generally, we can see that any corner of the map can be determined uniquely by how many *L*'s and *R*'s it takes to get there. If there are n total steps in the path to get to a corner and there are k *L*'s, then there are $n - k$ *R*'s. The number of paths to that corner is thus the number of combinations of k *L*'s and $n - k$ *R*'s. As is well known, this is the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In particular, the number of ways abracadabra can be spelled out in the diagram is $\binom{10}{5} = 252$. Thus, we use the binomial coefficients to label the corners, and it should be apparent that we are constructing Pascal's triangle. When looking at the binomial coefficient for a particular corner, remember that the n 's indicate the row of the triangle whereas the k 's count the number of lefts taken. For an excellent history of Pascal's triangle, see [3], which includes an English translation of Pascal's original treatise [7].

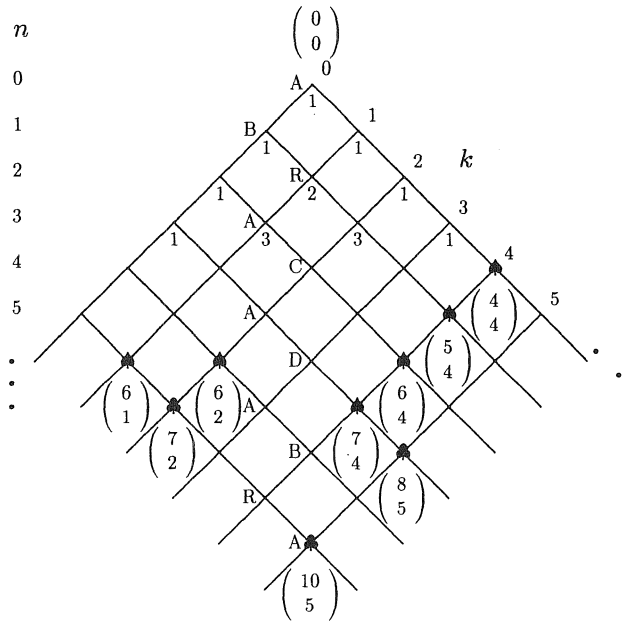


FIGURE 2. The Hockey Stick Identity gets its name from the shape of a blockwalking argument in Pascal's triangle.

Pascal's triangle is often created by first placing a 1 at each corner along the outside edges of the triangle. Second, an entry not on the edge is calculated by adding the two entries immediately above it. The top four rows generated using this method are shown in Figure 2, where the fourth row is 1 3 3 1. In terms of binomial coefficients, the second part of this construction is equivalent to the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1}$$

Surprisingly, it appears that Pólya never published a description of how blockwalking can be used to prove combinatorial identities—although it has been described elsewhere (e.g., [5], [13]). We introduce this technique by providing a blockwalking proof of (1). Partition the paths to $\binom{n}{k}$ depending on whether the last step is an L or an R . The number of paths with last step L is $\binom{n-1}{k-1}$ and the number of paths with last step R is $\binom{n-1}{k}$. Summing these gives (1). Conceptually, this is equivalent to setting up a *sieve* in the streets so that we must pass through the sieve to our destination and so that once we have passed through the sieve there is only one path to our final destination. Specifically, a sieve is defined to be a set of corners which partitions the paths to a particular corner into equivalence classes. For example, as shown in Figure 2, in order to get to the club at $\binom{7}{2}$, we must pass through one of the spades at $\binom{6}{1}$ or $\binom{6}{2}$. Also, there is only one way to get to $\binom{7}{2}$ from either of these spades. Thus, the corners indicated by these two spades form a sieve for the corner indicated by the club. (Note that identities involving products of binomial coefficients can be shown using sieves for which there is more than one path to the destination upon exiting a sieve.)

A different sieve leads us to the *Hockey Stick Identity*:

$$\sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1}. \tag{2}$$

Consider that for any path there must be a last L after which all other choices are R 's. Thus, we may partition paths to $\binom{n+1}{k+1}$ into classes depending on how many consecutive R 's are at the end of the path. The sum of the sizes of these classes thus equals the number of paths to $\binom{n+1}{k+1}$. This proves (2). Again, the equivalence classes can be associated with a sieve of corners. For example, the four spades from $\binom{4}{4}$ to $\binom{7}{4}$ form a sieve for $\binom{8}{5}$. This example also illustrates the use of the name *Hockey Stick Identity*. (This is also called the *Stocking Identity*. Does anyone know who first used these names?)

The following sections provide two distinct generalizations of the blockwalking technique. They are illustrated by proving distinct generalizations of the Hockey Stick Identity. We will be using the standard Hockey Stick Identity several times to prove these generalized forms. When we wish to do so by referencing its pictorial representation, we will refer to "summing the spades into the club."

2. MULTITIERED BLOCKWALKING

A somewhat obvious variation of the Hockey Stick Identity is to multiply the binomial coefficient inside the summation by i . Surprisingly, the author has not been able to find a reference for this variation (including [4] and [12]). We will give a generalized blockwalking argument to provide a closed form for this variation which we will call the *Extended Hockey Stick Identity*:

$$\sum_{i=k}^n i \binom{i}{k} = n \binom{n+1}{k+1} - \binom{n+1}{k+2}. \tag{3}$$

As illustrated in Figure 3, stack copies of Pascal's triangle on top of each other in tiers. The multitiered blockwalking technique follows these rules:

1. On each tier, you may walk blocks in the normal 2-dimensional fashion.
2. Your starting point may be at the apex on any tier.
3. At the end of a walk on your starting tier, you may step into an elevator and be raised a given number of tiers.
4. After having been elevated, you are at your final destination.

Any such path may be considered a series of L 's and R 's followed by some number of U 's (ups). (The alternative interpretation, which allows moving up at *any* point along a path, will be discussed under multinomial blockwalking in the next section.)

As with the standard single-tiered case, we label a corner with the number of paths to that corner. Using the specified rules, the $\binom{n}{k}$ corner on Tier t counts $t \binom{n}{k}$ paths; $\binom{n}{k}$ paths for each of t tiers. Figure 3 shows the path $LRLRLUUU$ which starts on Tier 3 and ends at $6 \binom{7}{3}$. (For pictorial convenience, Figures 3 and 4 have the k 's count R 's instead of L 's. This is symmetric to the standard representation of Pascal's triangle in Figure 2.)

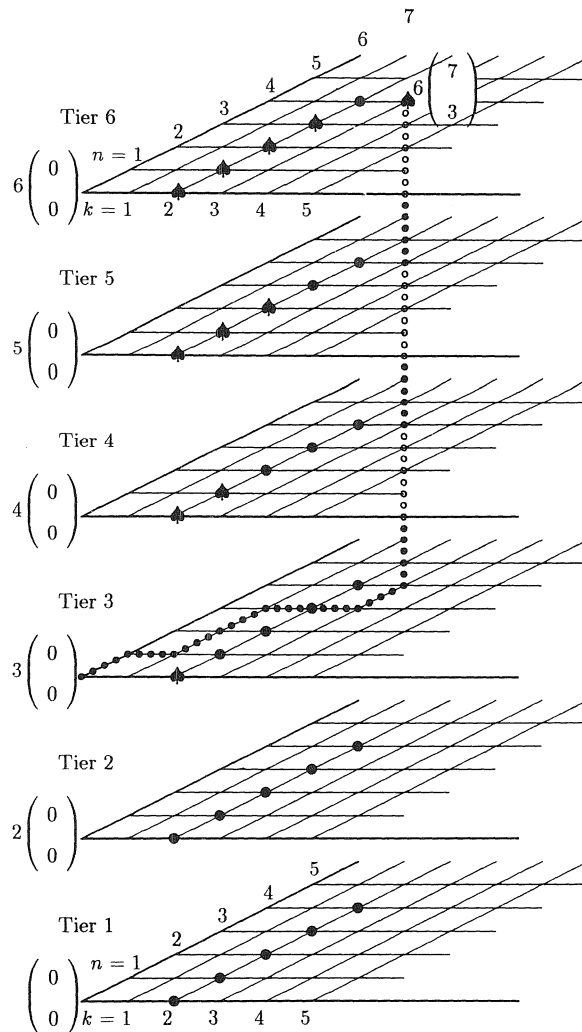


FIGURE 3. Multitiered Blockwalking

We can partition the paths to $n \binom{n+1}{k+1}$ into equivalence classes based on how many U 's are at the end and how many consecutive L 's precede the U 's. We illustrate this with $n = 6$ and $k = 2$ in Figure 3. The sieve for $6 \binom{7}{3}$ is shown using spades and filled circles. This illustrates:

$$n \binom{n+1}{k+1} = n \sum_{i=k}^n \binom{i}{k} = \sum_{i=k}^n n \binom{i}{k}. \quad (4)$$

By splitting the corners in the sieve appropriately, we illustrate:

$$\sum_{i=k}^n n \binom{i}{k} = \sum_{i=k}^n i \binom{i}{k} + \sum_{i=k}^n (n-i) \binom{i}{k}. \tag{5}$$

The filled circles in Figure 3 represent the first summation on the right-hand side of (5) and the spades represent the second summation. By rearranging (5) and applying (4), we get

$$\sum_{i=k}^n i \binom{i}{k} = n \binom{n+1}{k+1} - \sum_{i=k}^n (n-i) \binom{i}{k}. \tag{6}$$

In order to complete the proof of (3), we need to find a closed form for the subtrahend in (6).

A copy of Pascal's triangle can be formed by slicing a plane through the tiers as shown in Figure 4. That is, consider the plane that slices in front of the circles and through the last spade on each tier. This newly formed Pascal triangle consists of one street from each tier starting on Tier $t = k + 1$. In the example, the corner labeled $3 \binom{0}{0}$ is the apex of the new slicing Pascal triangle. Other nodes on this new Pascal triangle are labeled the same as they originally were except that the binomial coefficient is not multiplied by the tier number.

Using the standard Hockey Stick Identity on the spades on each tier (i.e., sum the spades into the clubs in Figure 4), we see that

$$\sum_{i=k}^n (n-i) \binom{i}{k} = \sum_{i=k}^{n-1} \sum_{j=k}^i \binom{j}{k} = \sum_{i=k}^{n-1} \binom{i+1}{k+1}.$$

Applying the standard Hockey Stick Identity on the slicing plane (i.e., summing the clubs into the circle at $\binom{7}{4}$ in Figure 4) gives

$$\sum_{i=k}^n (n-i) \binom{i}{k} = \sum_{i=k}^{n-1} \binom{i+1}{k+1} = \binom{n+1}{k+2}. \tag{7}$$

Combining (6) with (7) gives (3).

In personal correspondences to the author William Webb proved (3) using the "Snake Oil Method" of [14] and Bruce Berndt proved (3) using Vandermonde's theorem involving hypergeometric series (see [1]). However, a telescoping series gives the *Generalized Extended Hockey Stick Identity*:

$$\begin{aligned} \sum_{i=k}^n i^p \binom{i}{k} &= k^p \sum_{i=k}^n \binom{i}{k} + ((k+1)^p - k^p) \sum_{i=k+1}^n \binom{i}{k} + \dots + (n^p - (n-1)^p) \sum_{i=n}^n \binom{i}{k} \\ &= k^p \binom{n+1}{k+1} + \sum_{i=k+1}^n (i^p - (i-1)^p) \left[\binom{n+1}{k+1} - \binom{i}{k+1} \right]. \end{aligned}$$

Note that $i^p - (i-1)^p$ is a polynomial of degree $p-1$. Thus, the p^{th} case can be determined recursively.

Let us consider this generalized form in terms of multitiered blockwalking. When $p = 1$, the structure used to slice through the multitiered sieve is a line (between the spades and circles). Similar (though much more complicated) arguments could be provided for the p^{th} case by slicing the sieve with a curve of the form i^p .

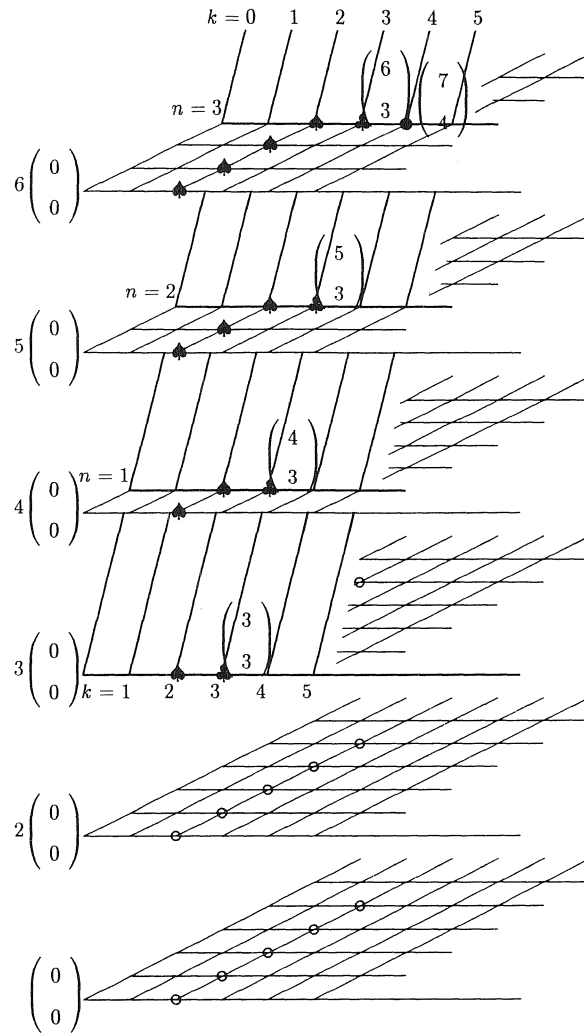


FIGURE 4. Slicing through the tiers and between the spades and circles.

3. MULTINOMIAL BLOCKWALKING

Perhaps a more natural generalization of the blockwalking technique to 3 dimensions is to allow moving up at *any* point along the path rather than just at the end. Also, it seems more natural to start at only one place—the origin. (Such a 3-dimensional version of the abracadabra concept is presented in [6].) This is easily generalized to j dimensions. That is, when blockwalking in j dimensions, not only can we walk along the standard 2 dimensions, but we can invoke our magic abracadabra amulet to take steps in all other $j-2$ dimensions. The grid we are walking on thus becomes the nonnegative j -dimensional lattice with corners labeled by multinomials

$$\binom{n}{n_1, n_2, \dots, n_j} = \frac{n!}{n_1! n_2! \dots n_j!}, \text{ where } n_1 + n_2 + \dots + n_j = n.$$

This is also called a *Pascal hyperpyramid* or a *Pascal polytope*. Generalizations of Pascal's original results to hyperpyramids can be found in [10] and [11]. An excellent survey and an extensive bibliography of Pascal hyperpyramids can be found in [2].

Figure 5 illustrates blockwalking in 5 dimensions:

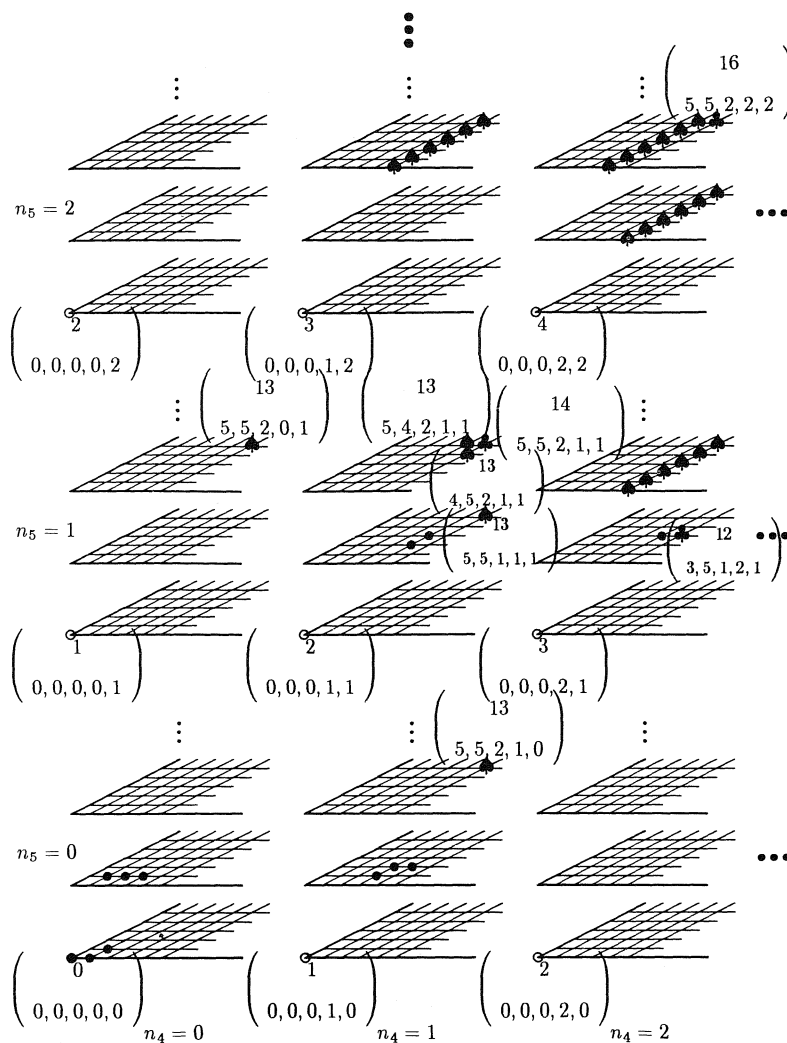


FIGURE 5. Blockwalking in 5 dimensions

Here, each tier represents a 2-dimensional blockwalk; tiers stacked on top of each other represent the 3rd dimension; the 4th dimension is represented by a row of 3-dimensional tiers going from left to right; the 5th dimension is represented by copies of the 4-dimensional rows stacked one above another. In 5 dimensions, we consider combinations of L 's, R 's, U 's, H 's

(hyperspaces), and W 's (warpdrives). The path $RLURRHLRWLHR$ is illustrated by circles in Figure 5. This path starts at the origin and ends at

$$\binom{12}{3, 5, 1, 2, 1}.$$

The recurrence relation (1) for Pascal's triangle generalizes for hyperpyramids:

$$\binom{n}{n_1, n_2, \dots, n_j} = \sum_{k=1}^j \binom{n-1}{n_1, \dots, n_k - 1, \dots, n_j}.$$

The blockwalking argument also generalizes. That is, partition the paths depending on what the last step is. For example, in Figure 5, the club at

$$\binom{14}{5, 5, 2, 1, 1}$$

has a sieve of five spades—each of which is one step away in one of five dimensions.

This brings us to the *Multinomial Hockey Stick Identity*:

$$\binom{n}{n_1, n_2, \dots, n_j} = \sum_{k=2}^j \sum_{i=n_2}^{n_1+n_2} \binom{i+n_3+n_4+\dots+n_j-1}{i-n_2, n_2, n_3, \dots, n_k-1, \dots, n_j}.$$

The sieve for this identity has $j-1$ copies of the sieve for the standard Hockey Stick Identity. This is illustrated in Figure 5 (with $j=5$) by the four rows of spades that form a sieve for the club at

$$\binom{16}{5, 5, 2, 2, 2}.$$

4. COMMENTS

The beauty of Pólya's blockwalking is that it provides a geometric interpretation of algebraic equations and, in so doing, gives a way of visualizing the concepts involved. The attempt here has been to utilize this idea to provide geometric and visual aids for more complex equations and higher dimensions. In developing the presentation of this visualization, the author conceived of the 2-dimensional representation of 5 dimensions shown in Figure 5. Although the 4-dimensional tesseract is often drawn in 2 dimensions, the author has never seen an attempt to represent 5 dimensions on a flat surface. The final comments consider how the representation in Figure 5 can be generalized to more dimensions.

The basic concept is to represent the n^{th} dimension by making copies of the $(n-1)^{\text{st}}$ dimension. Thus, to visualize the 6th dimension, we make copies of Figure 5, say, by stacking pages on top of each other. Notice that this gives us a 3-dimensional array of 3-dimensional arrays. We can visualize further dimensions by considering larger and larger 3-dimensional arrays. For example, we can consider our 6-dimensional stack of pages as a ream of paper. Obtain the next 3 dimensions by considering a 3-dimensional array of reams. Continuing in this fashion, call the

array of reams a box; consider a 3-dimensional array of boxes, call this a pallet; consider a 3-dimensional array of pallets; etc. The structure just described, up through a 3-dimensional array of pallets, represents a 15-dimensional nonnegative lattice. The limiting factor in this method is our ability to establish a mnemonically reasonable ordering of containers to describe each level of 3-dimensional arrays. This approach can be compared to a Mandelbrot set in that we are maintaining the same visual picture only at larger and larger scales.

The visualization just described is aided by the discrete nature of the lattice (i.e., the blocks of the blockwalking) and by restricting the picture to the nonnegative "hyperquadrant" of the lattice. In order to eliminate the discrete nature of our picture, instead of making discrete copies of the $(n-1)$ -dimensional representation, move the $(n-1)$ -dimensional representation in a continuous motion. To remove the restriction of only visualizing the nonnegative hyperquadrant of n -space, consider 2^n copies of the hyperquadrant to represent the 2^n hyperquadrants that exist in n -space.

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