DYNAMICS OF THE ZEROS OF FIBONACCI POLYNOMIALS

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1. INTRODUCTION

The Fibonacci polynomials are defined by the recursion relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \tag{1}$$

with the initial values $F_1(x) = 1$ and $F_2(x) = x$. When $x = 1$, $F_n(x)$ is equal to the n^{th} Fibonacci number, F_n. The Lucas polynomials, $L_n(x)$ obey the same recursion relation, but have initial values $L_1(x) = x$ and $L_2(x) = x^2 + 2$.

Explicit expressions for the zeros of the Fibonacci and Lucas polynomials have been known for some time ([1], [2]). The zeros of $F_{2n}(x)$ are at the points

$$\pm 2i \sin\frac{k\pi}{2n}, \quad k = 0, 1, \ldots, n-1. \tag{2}$$

The zeros of the odd polynomials $F_{2n+1}(x)$ are at

$$\pm 2i \sin\left[\left(\frac{2k+1}{2n+1}\right)\cdot\frac{\pi}{2}\right], \quad k = 0, 1, \ldots, n-1. \tag{3}$$

Similarly, for the Lucas polynomials, the zeros of $L_{2n}(x)$ are at

$$\pm 2i \sin\left[\left(\frac{2k+1}{2n}\right)\cdot\frac{\pi}{2}\right], \quad k = 0, 1, \ldots, n-1, \tag{4}$$

and the zeros of $L_{2n+1}(x)$ are at

$$\pm 2 \sin\frac{k\pi}{2n+1}, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

With a view toward finding clues to obtaining similar analytic expressions for the zeros of the Tribonacci polynomials [3] and other generalizations of the $F_n(x)$, it is of interest to study the properties of the above expressions in more detail, looking for patterns that may generalize. In what follows, it will be shown that the zeros of each $F_n(x)$ and $L_n(x)$ satisfy a number of relations among themselves, many of which can be derived without any knowledge of the explicit formulas given above. The results presented here divide into two parts: in §2, expressions for the elementary symmetric polynomials of the zeros of each polynomial are derived. Then in §3, the zeros are described in terms of points on the trajectories of a dynamical system. In §4, some comments are made regarding the generalization of these results to the Tribonacci case.

2. SYMMETRIC POLYNOMIALS

Consider the elementary symmetric polynomials $\sigma_j(x_1, x_2, \dots, x_m)$ over the set $x_1, x_2, \dots, x_m$, where $0 \le j \le m$. These polynomials are defined by the relation

$$\prod_{k=1}^{m}(t+x_k) = \sum_{j=0}^{m}\sigma_j(x_1, \dots, x_m)\cdot t^{m-j}. \tag{6}$$

Clearly, σ_j is a polynomial of order j in its m arguments. Note that by multiplying out the left-hand side and comparing powers of t on each side, we can write the σ_j as

$$\sigma_j(x_1, \dots, x_m) = \sum_{l_1=0}^{m-1}\sum_{l_2=l_1}^{m-1}\cdots\sum_{l_j=l_{j-1}}^{m-1}\prod_{i=1}^{j}x_{l_i}. \tag{7}$$

The idea in the following theorems is to derive general formulas for the symmetric polynomials over the zeros, using the following algebraic representations of the Fibonacci and Lucas polynomials [2]:

$$F_n(x) = \sum_{k=0}^{\left[\frac{(n-1)}{2}\right]}\binom{n-k-1}{k}x^{n-2k-1}, \tag{8}$$

and

$$L_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}, \tag{9}$$

where $[p]$ means the greatest integer less than or equal to p.

First, let us consider the $F_n(x)$. Since the zeros of $F_n(x)$ are pure imaginary and come in complex conjugate pairs, we will concentrate on their magnitudes. Thus, for the even polynomials $F_{2n}(x)$, denote the zeros by

$$x_0 = 0, \quad \pm ix_k, \quad k = 1, 2, \dots, n-1, \tag{10}$$

with $x_k > 0$ for $k > 0$. As for the odd polynomials, $F_{2n+1}(x)$, denote the zeros by

$$\pm ix_k, \quad k = 0, 1, \dots, n-1, \tag{11}$$

where

$$x_k = 2\sin\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{2}\right), \quad k = 0, 1, \dots, n-1. \tag{12}$$

Theorem 1: The j^{th} symmetric polynomial over the squares of the zeros of $F_{2n}(x)$ is given by

$$\sigma_j(x_1^2, \dots, x_{n-1}^2) = \binom{2n-j-1}{j}. \tag{13}$$

Proof: Clearly, since the zeros are of the form given in formula (10) above, the $F_{2n}(x)$ can be factored as follows:

$$F_{2n}(x) = x\prod_{k=1}^{n-1}(x-ix_k)(x+ix_k). \tag{14}$$

We can then regroup this expression in the following manner:

$$\begin{aligned}
F_{2n}(x) &= x\prod_{k=1}^{n-1}(x^2+x_k^2) \\
&= x\left\{x^{2n-2}+x^{2n-4}\sum_{k=1}^{n-1}x_k^2+x^{2n-6}\sum_{j\neq k}x_j^2x_k^2+x^{2n-8}\sum_{j\neq k\neq l}x_j^2x_k^2x_l^2+\cdots+\prod_{k=1}^{n-1}x_k^2\right\} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}x^{2n-2j-1}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^2\right\} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}\sigma_j(x_1^2,\ldots,x_{n-1}^2)x^{2n-2j-1}.
\end{aligned} \tag{15}$$

But we also know that

$$\begin{aligned}
F_{2n}(x) &= \sum_{j=0}^{n-1}\binom{2n-j-1}{j}x^{2n-2j-1} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}\binom{2n-j-1}{j}x^{2n-2j-1}.
\end{aligned} \tag{16}$$

Setting the right-hand sides of equations (15) and (16) equal and equating the coefficient of each power of x, we arrive at the desired result. □

Alternatively, this theorem and those that follow can be proved by applying standard trigonometric identities to the explicit formulas for the zeros that were given in equations (2) through (5).

Corollary 1: The zeros of the even polynomials $F_{2n}(x)$ satisfy the following relations for fixed n:

(i) $\prod_{k=1}^{n-1}x_k^2=n$,

(ii) $\sum_{k=0}^{n-1}x_k^2=2(n-1)$.

Proof: These follow immediately by setting $j=1$ and $j=n-1$, respectively, in the previous theorem. □

Turning now to the odd Fibonacci polynomials, the following result can be quickly proved in the same manner.

Theorem 2: The j^{th} symmetric polynomial over the zeros of $F_{2n+1}(x)$ is given by the expression

$$\sigma_j(x_1^2,\ldots,x_{n-1}^2)=\binom{2n-j}{j}. \tag{17}$$

Corollary 2: For fixed n, the zeros of $F_{2n+1}(x)$ satisfy the following relations:

(i) $\prod_{k=0}^{n-1}x_k^2=1$,

(ii) $\sum_{k=0}^{n-1}x_k^2=2n-1$.

Proof: In the previous theorem, set $j=1$ to obtain (i) and $j=n$ to obtain (ii). □

Theorems 1 and 2 have been checked numerically for the polynomials $F_1(x)$ through $F_{13}(x)$. The corollaries have been checked numerically for all values from $n=1$ to $n=20$, as well as for selected values up to $n=1000$. The numerical results show perfect agreement with the results predicted here.

3. THE DYNAMICS OF THE ZEROS

The goal here is to obtain the zeros of $F_n(x)$ as iterates of some function (independent of n) which maps the zeros of $F_{n-1}(x)$ to the zeros of $F_n(x)$. This procedure is complicated by the fact that the number of zeros increases with increasing n, but that will be dealt with below by breaking up the zeros into one parameter families, with n as the parameter. A second parameter, m, will distinguish one family from the next. Although the recursion relations derived below contain no information that is not already implicitly contained in formulas (2) through (5), it provides a different perspective on this information. Also, this recursion relation method can provide an algorithm that may be more efficient than other methods for numerical calculations of zeros for other classes of polynomials when the zeros do not have such simple analytic formulas.

As in earlier sections, rather than dealing directly with the zeros, $\pm ix_j$, we will deal only with their magnitudes, x_j. However, for our purposes here, it is convenient to alter our notation slightly. For a fixed value of n, label the magnitudes of the zeros in decreasing order as follows: $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$. The superscript labels the polynomial of which it is a zero, and the subscript labels the relative size of the zero. Using this ordering, $x_{n/2}^{(n)}$ always vanishes for even n. For a generic zero $x_m^{(n)}$ of $F_n(x)$, we will call m the *row number* of the zero, for reasons that will become apparent later. The idea is to find a function $f_m : \Re \to \Re$, independent of n, such that $f_m(x_m^{(n)}) = x_m^{(n+1)}$. As we will see below, the zeros $x_m^{(n)}$ for all n will then be obtainable by applying the appropriate f_m to the initial value $x=0$, and then iterating a certain number of times. The main result is Theorem 3 below.

Theorem 3: For all $n \geq 2$, the zero in the m^{th} row of $F_{n+1}(x)$ is related to the zero in the m^{th} row of $F_n(x)$ by the following mapping:

$$x_m^{(n+1)} = \frac{2}{\sqrt{1+\alpha_m(x_m^{(n)})}}, \tag{18}$$

where

$$\alpha_m(x) = \tan^2\left\{ m\pi \left[\frac{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}} + 2\pi K_1}{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}} + 2\pi K_1 + m\pi} \right] - 2\pi K_2 \right\}, \tag{19}$$

where K_1 and K_2 are a pair of integer constants.

Proof: Assume for the sake of definiteness that n is even. [If n is odd, the proof proceeds in an identical manner, except that the roles of equations (2) and (3) are reversed.] Referring to equations (2) and (3), the integer k is related to the row number m by $k = n - m$, so that these equations tell us that the zeros of $F_n(x)$ and $F_{n+1}(x)$ are at

$$x_m^{(n)} = 2\sin\left(\frac{n-m}{2n}\right)\pi = 2\cos\left(\frac{m\pi}{2n}\right), \tag{20}$$

$$x_m^{(n+1)} = 2\sin\left[\left(\frac{2n+1-2m}{2n+1}\right)\frac{\pi}{2}\right] = 2\cos\left(\frac{m\pi}{2n+1}\right), \tag{21}$$

where we have used the fact that $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$.

Now, note that

$$\exp\left(\frac{im\pi}{2n}\right) = \cos\frac{m\pi}{2n} + i\sin\frac{m\pi}{2n} = \frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}. \tag{22}$$

Taking the natural logarithm of the last equation gives

$$\frac{im\pi}{2n} = \ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] + 2i\pi K_1, \tag{23}$$

where K_1 is an integer that specifies which branch of the Riemann surface is used to evaluate the logarithm. Now, solve for $2n$:

$$2n = \frac{im\pi}{\ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] + 2\pi i K_1}. \tag{24}$$

Repeating the procedure of the previous paragraph, but this time applying it to $\exp\left(\frac{im\pi}{2n+1}\right)$, we find

$$2n+1 = \frac{im\pi}{\ln\left[\frac{x_m^{(n+1)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n+1)}}{2}\right)^2}\right] + 2\pi i K_2}. \tag{25}$$

where, again, K_2 is an integer constant.

Substituting equation (24) into equation (25) yields

$$\frac{im\pi}{\ln\left[\frac{x_m^{(n+1)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n+1)}}{2}\right)^2}\right] 2\pi i K_2} = \frac{im\pi}{\ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] 2\pi i K_1} + 1. \tag{26}$$

This result can be simplified. Note that, for any variable y such that $-2 \le y \le 2$, we can define a pair of polar coordinates (r, θ) via

$$\frac{y}{2} + i\sqrt{1-\left(\frac{y}{2}\right)^2} = r\exp i\theta. \tag{27}$$

Clearly,

$$r = 1, \quad \theta = \tan^{-1}\sqrt{\frac{4-y^2}{y^2}}. \tag{28}$$

Taking the natural logarithm of equation (27),

$$\begin{aligned} \ln\left[\frac{y}{2}+i\sqrt{1-\left(\frac{y}{2}\right)^2}\,\right] &= \ln(r\exp i\theta) \\ &= \ln r + i\theta \\ &= i\tan^{-1}\sqrt{\frac{4-y^2}{y^2}}\,. \end{aligned} \tag{29}$$

Finally, applying equation (29) to both sides of formula (26), and then solving for $x_m^{(n+1)}$ gives the desired result. □

Note that two integer constants, K_1 and K_2, appear in this result. From examining equation (19), it is clear that K_1 is completely arbitrary; changing its value will simply change the argument of the tangent by a multiple of 2π. Because of the periodicity of the tangent, the value of K_1 has no effect on the results and will henceforth be set to zero.

The second constant, K_2, enters into the proof in the same way but, curiously, its value *does* affect the positions of the zeros. Theorem 3 has been checked numerically by using it to predict the first 40 zeros for all cases from $m=1$ to $m=10$. In each case, the theorem gives the correct results, provided that K_2 is set equal to zero. Allowing K_2 to have nonzero values seems to lead to interesting effects; these are currently under investigation. But in the remainder of this paper, we will set $K_2=0$ (or, in other words, we will restrict ourselves to the principal branches of all logarithms), since this is the case that gives the correct zeros for the Fibonacci polynomials.

Theorem 3 tells us that families of $x_m^{(n)}$ with fixed m form trajectories of a dynamical system, with n playing the role of a discrete time variable. Points are moved along each trajectory by repeated iteration of the function $f_m(x)=2/\sqrt{1+\alpha_m(x)}$. This situation is illustrated in Figure 1.

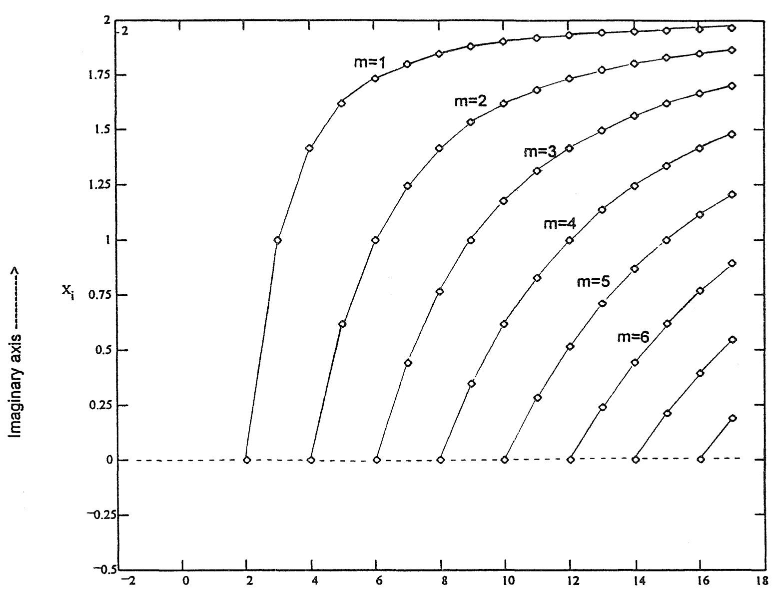

FIGURE 1: The Zeros of the Fibonacci Polynomials
(Only zeros with nonnegative imaginary part are shown)

It can be seen that the integer m labels how many rows the trajectory is from the outside of the diagram. It is also clear that each trajectory begins at a root of the form $x_m^{(n)} = x_m^{(2m)} = 0$. Since each iteration of f_m increases n by one, and since each trajectory starts at an initial value of $n_0 = 2m$, it takes $n-2m$ iterations to reach a fixed final value of n. As a consequence, we have the following corollary.

Corollary 3: The zero of $F_n(x)$ with row number m can be written as

$$x_m^{(n)} = f_m^{(n-2m)}(0), \tag{30}$$

where $f_m^{(j)}$ means the j^{th} iterate of f_m.

Some observations can be made about this result. First, it is clear from the form of $f_m(x)$ that each trajectory approaches an attracting fixed point situated at $x^{(\infty)} = 2$. This implies that as $n \to \infty$, $x_m^{(n)} \to 2$, for all m.

Second, a similar result is easily proved for the zeros of the Lucas polynomials by using the same method. In the Lucas case, we still have $f_m(x) = 2/\sqrt{1+\alpha_m(x)}$, but now the form of α_m changes:

$$\alpha_m(x) = \tan^2\left[(2m-1)\pi \cdot \frac{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}}}{(2m-1)\pi + 2\tan^{-1}\sqrt{\frac{4-x^2}{x^2}}}\right]. \tag{31}$$

Here, we have again set $K_1 = K_2 = 0$. There is one complication arising here that did not occur in the Fibonacci case: iteration of the above function does not simply carry us along the m^{th} row. Instead, the trajectory jumps back and forth between two adjacent rows. More specifically, repeated use of α_m will give us the zeros in the m^{th} row for $L_{2n}(x)$ and those in row $m+1$ for $L_{2n+1}(x)$. This occurs because m enters the expressions for the even and odd zeros in the same manner for the Fibonacci case [compare the numerators of the last expressions in equations (20) and (21)], while in the corresponding expres-sions for the Lucas zeros, it enters through a factor of $(2m+1)$ in one case and $(2m-1)$ in the other. Although the trajectory now alternates rows, we still recover all of the zeros as we run over differing values of m, as has been verified numerically.

Some observations can also be made about the properties of the $\alpha_m(x)$. For the Fibonacci case, define

$$\beta_m^{(n)} = \sqrt{\frac{4-(x_m^{(n)})^2}{(x_m^{(n)})^2}} \quad \text{and} \quad \alpha_m^{(n)} = \alpha_m(x_m^{(n)}) = \tan^2\left\{\frac{\tan^{-1}\beta_m^{(n)}}{\tan^{-1}\beta_m^{(n)} + \pi} \cdot m\pi\right\}.$$

Then we have the following propositions.

Proposition 1: For all m and n,

$$\beta_m^{(n)} = \sqrt{\alpha_m^{(n-1)}}. \tag{32}$$

Proof: We know that

$$\beta_m^{(n)} = \sqrt{\frac{4-(x_m^{(n)})^2}{(x_m^{(n)})^2}} = \sqrt{\frac{4-f_m^2(x_m^{(n-1)})}{f_m^2(x_m^{(n-1)})}}. \tag{33}$$

Substituting $f_m(x) = 2 / \sqrt{a + \alpha_m(x)}$ into this expression and simplifying the fraction quickly leads to equation (32). □

Proposition 2: The argument of the tangent in $\alpha_m^{(n)}$ is always a rational multiple of π. In other words, the quantity

$$\frac{\tan^{-1} \beta_m^{(n)}}{\tan^{-1} \beta_m^{(n)} + \pi} \tag{34}$$

is rational for all n and m.

Proof: We know [by equations (20) and (21) or, alternately, by equations (2) and (3)] that all of the zeros can be written in the form $x_m^{(n)} = 2\cos\left(\frac{p}{q}\pi\right)$ for some pair of integers p and q (depending on m and n). Substituting this expression into the definition of $\beta_m^{(n)}$, we find that $B_m^{(n)} = \tan\frac{p}{q}\pi$, or $\tan^{-1}\beta_m^{(n)} = \frac{p}{q}\pi$. Substituting this into the quantity in formula (34), we find that it equals $\frac{p}{p+1}$, which is clearly rational. □

Note that $\beta_m^{(n)}$ describes the tangent of an angle inscribed in a right triangle of hypotenuse equal to 2, and adjacent side of length $x_m^{(n)}$. The hypotenuse remains constant, while the adjacent side increases in length with increasing n or decreasing m. A deeper understanding of the geometric meanings of $\alpha_m^{(n)}$ and $\beta_m^{(n)}$ may help provide some insight into the properties of the zeros of the Tribonacci polynomials and other generalizations of the $F_n(x)$.

4. TRIBONACCI POLYNOMIALS

The Fibonacci and Lucas polynomials have been generalized in various ways. The simplest generalization is that of the Tribonacci polynomials, $T_n(x)$ (see [3]), which obey the relation

$$T_{n+3}(x) = x^2 T_{n+2}(x) + x T_{n+1}(x) + T_n(x), \tag{35}$$

with $T_0(x) = 0$, $T_1(x) = 1$, $T_2(x) = x^2$. The $T_n(x)$ are often written in terms of the trinomial coefficients $\binom{m}{j}_3$, which are defined implicitly by the following equation [3]:

$$T_n(x) = \sum_{j=0}^{\left[\frac{2}{3}(n-1)\right]} \binom{n-j-1}{j}_3 x^{2n-3j-2}. \tag{36}$$

While numerical work has been done concerning the zeros of the Tribonacci polynomials, explicit expressions for them are not known, so deriving formulas of the sort presented in §2 of this paper would be of interest, as they could provide valuable clues to the possible forms the zeros could have. Below is a theorem giving expressions for the symmetric polynomials of the Tribonacci zeros. Again, these results are easily verified numerically. The proofs are omitted, as they are identical to those of §2, except that equation (36) replaces equation (8).

The zeros of the Tribonacci polynomials form a set that is invariant under rotations in the complex plane by multiples of $2\pi/3$, so the zeros can be divided into three subsets: $\{x_i\}$, $\{x_i e^{2\pi/3}\}$, and $\{x_i e^{-2\pi/3}\}$, for an appropriate set of x_i.

Theorem 4:

(i) The zeros of $T_{3n+1}(x)$ have elementary symmetric polynomials of the form

$$\sigma_j(x_1^3, \ldots, x_{2n}^3) = (-1)^j \binom{3n-j}{j}_3. \tag{37}$$

(ii) The zeros of $T_{3n+2}(x)$ satisfy the following relation:

$$\sigma_j(x_1^3, \ldots, x_{2n}^3) = (-1)^j \binom{3n-j+1}{j}_3. \tag{38}$$

(iii) The zeros of $T_{3n}(x)$ satisfy the following relation:

$$\sigma_j(x_1^3, \ldots, x_{2n-1}^3) = (-1)^j \binom{3n-j-1}{j}_3. \tag{39}$$

By setting $j = 1$ in the above theorem, we have the following corollary.

Corollary 4:

(i) The zeros of $T_{3n+1}(x)$ satisfy $\sum_{k=1}^{2n} x_k^3 = -(3n-1)$.

(ii) The zeros of $T_{3n+2}(x)$ satisfy $\sum_{k=1}^{2n} x_k^3 = -3n$.

(iii) The zeros of $T_{3n}(x)$ satisfy $\sum_{k=1}^{2n-1} x_k^3 = -(3n-2)$.

As for the results presented in §3 of this paper, their derivation depended on prior knowledge of the explicit formulas for the zeros of the $F_n(x)$. However, the logic could be reversed: *if* formulas analogous to the f_m could be found for the $T_n(x)$ by fitting functions to a few of the numerically known zeros, then explicit formulas for the positions of all the zeros could immediately be generated. Finding the f_m functions and finding the zeros are thus equivalent problems, but it could turn out that one form of the problem is easier than the other. Finding the f_m functions could be aided by further analysis of the geometrical content of the results of §3. and of how the geometry changes in the Tribonacci case.

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