ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The Fibonacci polynomials, $F_n(x)$, and the Lucas polynomials, $L_n(x)$, satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad F_0(x) = 0, \quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \quad L_0(x) = 2, \quad L_1(x) = x.$$

Also,

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \text{ and } L_n(x) = \alpha(x)^n + \beta(x)^n,$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$.

PROBLEMS PROPOSED IN THIS ISSUE

Today's column is all about Fibonacci and Lucas polynomials, $F_n(x)$ *and* $L_n(x)$, *which are defined above. For more information about Fibonacci polynomials, see Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII—An Introduction to Fibonacci Polynomials and Their Divisibility Properties," The Fibonacci Quarterly* **8.4** *(1970):407-420.*

B-842 ***Proposed by the editor***

Prove that no Lucas polynomial is exactly divisible by $x-1$.

B-843 ***Proposed by R. Horace McNutt, Montreal, Canada***

Find the last three digits of $L_{1998}(114)$.

B-844 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

If $a+b$ is even and $a > b$, show that $[F_a(x) + F_b(x)][F_a(x) - F_b(x)] = F_{a+b}(x)F_{a-b}(x)$.

B-845 ***Proposed by Gene Ward Smith, Brunswick, ME***

Show that, if m and n are odd positive integers, then $L_n(L_m(x)) = L_m(L_n(x))$.

B-846 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***

Show that

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!}$$

is an integer for all integral k. Generalize.

B-847 ***Proposed by Gene Ward Smith, Brunswick, ME***

Find the greatest common polynomial divisor of $F_{n+4k}(x)+F_n(x)$ and $F_{n+4k-1}(x)+F_{n-1}(x)$.

B-837 (corrected) ***Proposed by Joseph J. Koštál, Chicago IL***

Let

$$P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$$

and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by L_{999}.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

It Keeps on Growing

B-826 ***Proposed by the editor***
(Vol. 35, no. 2, May 1997)

Find a recurrence consisting of positive integers such that each positive integer n occurs exactly n times.

Solution

All solvers selected the monotone sequence $\langle a_n \rangle = 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, whose n^{th} term they found to be $a_n = \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor$.

The recurrences found were:

$$a_n = 1 + a_{n-a_{n-1}};$$ ***L. A. G. Dresel***

$$a_n = a_{n-1} + \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor - \left\lfloor \frac{1+\sqrt{8n-15}}{2} \right\rfloor;$$ ***Gerald A. Heuer***

$$a_{n+1} = a_n + \left\lfloor \frac{1}{2n} \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \right\rfloor;$$ ***H.-J. Seiffert***

$$a_n = 1 + a_{\lfloor (2n+3-\sqrt{8n+9})/2 \rfloor};$$ ***Reginald H. McNutt***

$$a_{n+1} = a_n + \begin{cases} 1, & \text{if } n \text{ is triangular,} \\ 0, & \text{otherwise;} \end{cases}$$ ***Paul S. Bruckman***

each with initial condition $a_1 = 1$.

A Simple Third-Order Recurrence

B-827 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland (Vol. 35, no. 2, May 1997)***

Find a solution to the recurrence

$$A_{n+3} = A_n - 2A_{n+2},\ A_0 = 0,\ A_1 = 1,\ A_2 = -2,$$

in terms of F_n and L_n.

Solution by Graham Lord, Princeton, NJ

That $A_n = (-1)^{n-1}(F_{n+2} - 1)$ satisfies the recurrence is verified by substitution:

$$\begin{aligned} A_{n+3} + 2A_{n+2} &= (-1)^{n+2}(F_{n+5} - 1) + 2(-1)^{n+1}(F_{n+4} - 1) \\ &= (-1)^{n+2}(F_{n+5} - 2F_{n+4}) - (-1)^{n+2} - 2(-1)^{n+1} \\ &= (-1)^{n+2}(F_{n+3} - F_{n+4}) + (-1)^{n+1} - 2(-1)^{n+1} \\ &= (-1)^{n+1}F_{n+2} - (-1)^{n+1} \\ &= (-1)^{n+1}(F_{n+2} - 1) \\ &= (-1)^{n-1}(F_{n+2} - 1) \\ &= A_n. \end{aligned}$$

Also solved by Mohammad K. Azarian, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Gerald A. Heuer, Harris Kwong, Bob Prielipp, Maitland A. Rose, James A. Sellers, H.-J. Seiffert, I. Strazdins, and the proposer.

Semi Fibonacci

B-828 ***Proposed by Piero Filipponi, Rome, Italy (Vol. 35, no. 2, May 1997)***

For n a positive integer, prove that

$$\sum_{r=0}^{\lfloor \frac{n-1}{4} \rfloor} \binom{n-1-2r}{2r}$$

is within 1 of $F_n/2$.

Solution by H.-J. Seiffert, Berlin, Germany

Let n be a positive integer. It is well known ([2], p. 50) that

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} = F_n. \tag{1}$$

The formula ([1], p. 33)

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} (2\cos x)^{n-1-2k} = \frac{\sin nx}{\sin x}$$

when letting $x = \pi/3$ gives

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} = \frac{2}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right), \tag{2}$$

since $\cos(\pi/3)=1/2$ and $\sin(\pi/3)=\sqrt{3}/2$. Adding equations (1) and (2) and dividing the resulting equation by 2 yields

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1+(-1)^k}{2}\binom{n-1-k}{k}=\frac{1}{2}F_n+\frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right)$$

or, equivalently,

$$\sum_{k=0}^{\lfloor (n-1)/4\rfloor}\binom{n-1-2r}{2r}=\frac{1}{2}F_n+\frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Thus, the desired sum differs from $F_n/2$ by at most $\frac{1}{\sqrt{3}}\sin(\frac{n\pi}{3})$, which is less than 1.

The proposer also found the corresponding result for Lucas numbers:

$$\sum_{k=0}^{\lfloor n/4\rfloor}\frac{n}{n-2r}\binom{n-2r}{2r}=\begin{cases}(L_n+2)/2, & \text{if } n\equiv 0 \pmod 6,\\ (L_n+1)/2, & \text{if } n\equiv \pm 1 \pmod 6,\\ (L_n-1)/2, & \text{if } n\equiv \pm 2 \pmod 6,\\ (L_n-2)/2, & \text{if } n\equiv 3 \pmod 6.\end{cases}$$

References

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products.* 5th ed. San Diego, CA: Academic Press, 1994.
2. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by David M. Bloom, Paul S. Bruckman, Leonard A. G. Dresel, Indulis Strazdins, and the proposer.

Powers of 2

B-829 ***Proposed by Jack G. Segers, Liège, Belgium***
(Vol. 35, no. 2, May 1997)

For n a positive integer, let $P_n=F_{n+1}F_n$, $A_n=P_{n+1}-P_n$, $B_n=A_n-A_{n-1}$, $C_n=B_{n+1}-B_n$, $D_n=C_n-C_{n-1}$, and $E_n=D_{n+1}-D_n$. Show that $|P_n-B_n|$, $|A_n-C_n|$, $|B_n-D_n|$, and $|C_n-E_n|$ are successive powers of 2.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We generalize the result as follows. Define an array of integers $S_{i,n}$ by $S_{0,n}=F_{n+1}F_n$, and, for $k\geq 1$,

$$S_{2k-1,n}=S_{2k-2,n+1}-S_{2k-2,n},$$
$$S_{2k,n}=S_{2k-1,n}-S_{2k-1,n-1}.$$

Note that $S_{0,n}=P_n$ and $S_{i,n}$, $1\leq i\leq 5$, equals A_n, B_n, C_n, D_n, and E_n, respectively. We shall prove, by induction on i, that

$$S_{i,n}-S_{i+2,n}=(-1)^{n-1+\lceil i/2\rceil}2^i.$$

We have

$$S_{1,n}=S_{0,n+1}-S_{0,n}=F_{n+2}F_{n+1}-F_{n+1}F_n=F_{n+1}(F_{n+2}-F_n)=F_{n+1}^2.$$

It follows that

$$S_{0,n} - S_{2,n} = F_{n+1}F_n - (F_{n+1}^2 - F_n^2) = F_n^2 - F_{n+1}(F_{n+1} - F_n) = F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1},$$

hence the assertion holds for $i = 0$. In general, assume it holds for some $i \geq 0$. If $i+1$ is odd, then

$$\begin{aligned} S_{i+1,n} - S_{i+3,n} &= (S_{i,n+1} - S_{i,n}) - (S_{i+2,n+1} - S_{i+2,n}) \\ &= (S_{i,n+1} - S_{i+2,n+1}) - (S_{i,n} - S_{i+2,n}) \\ &= (-1)^{n+\lceil i/2 \rceil} 2^i - (-1)^{n-1+\lceil i/2 \rceil} 2^i \\ &= (-1)^{n+\lceil i/2 \rceil} 2^{i+1} \\ &= (-1)^{n-1+\lceil (i+1)/2 \rceil} 2^{i+1}. \end{aligned}$$

The induction is completed by proving the case of even $i+1$ in a similar manner. Therefore, the absolute differences stated in the problem are 1, 2, 4, and 8, respectively.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Graham Lord, Bob Prielipp, H.-J. Seiffert, and the proposer.

Offset Entries

B-830 ***Proposed by Al Dorp, Edgemere, NY (Vol. 35, no. 2, May 1997)***

(a) Prove that, if $n = 84$, then $(n+3) | F_n$.

(b) Find a positive integer n such that $(n+19) | F_n$.

(c) Is there an integer a such that $n + a$ never divides F_n?

Solution by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY

We use the following results:

Result 1 ([1], p. 37): If $d | n$, then $F_d | F_n$.

Result 2 ([1], p. 44): Every positive integer a divides some Fibonacci number F_n $(n > 0)$.

Result 3 ([2], p. 21): If $a | n$ and $b | n$, where a and b are relatively prime, then $ab | n$.

Result 4 ([2], p. 24): If a and b are relatively prime positive integers, then the arithmetic progression $\langle an + b \rangle$, $n = 1, 2, 3, \ldots$ contains infinitely many primes (Dirichlet's Theorem).

Result 5 ([3], p. 79): If the prime p is of the form $5t \pm 1$, then $p | F_{p-1}$.

(a) Since $3 | F_4$ and $29 | F_{14}$, we must have $3 | F_{84}$ and $29 | F_{84}$ by result 1. Thus, $87 | F_{84}$ by result 3.

(b) The integer $n = 2052 = 19 \cdot 108$ meets the conditions of part (b). For $19 | F_{18}$ and $109 | F_{27}$, so $19 | F_{19 \cdot 109}$ and $109 | F_{19 \cdot 109}$ by result 1. Thus, $(19 \cdot 108 + 19) | F_{19 \cdot 109}$ by result 3.

(c) The answer to part (c) is that, if a is any integer, then there must be a positive integer n such that $(n+a) | F_n$. For $a = 0$, $n = 5$ works. For $a < 0$, $n = 1 - a$ works.

If $a > 0$, there must be a positive integer b such that $a | F_b$ by result 2. By result 4, the arithmetic progression $10b+1$, $20b+1$, $30b+1$, ... contains infinitely many primes, so there exists a prime $p = 10kb + 1$ such that $p > a$. Since $p > a$, it must be relatively prime to a. Since $p \equiv 1$ (mod 10), p divides $F_{p-1} = F_{10kb}$ by result 5. Thus, $p | F_n$, where $n = 10kba$ by result 1.

Likewise, $a|F_b$ implies $a|F_n$ by result 1. Finally, $n+1=(10kb+1)a=pa$, which divides F_n by result 3.

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.
2. Don Redmond. *Number Theory: An Introduction.* New York: Marcel Dekker, 1996.
3. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

For part (b), Bloom found $n=19\cdot 108$ and Bruckman found $n=19\cdot 180$. Dresel removed the "0", finding that $n=19\cdot 18$ satisfies part (b).

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, and the proposer.

Minimal Polynomial

B-831 ***Proposed by the editor***
(Vol. 35, no. 3, August 1997)

Find a polynomial $f(x,y)$ with integer coefficients such that $f(F_n, L_n)=0$ for all integers n.

Solution

All solvers came up with

$$f(x,y)=(y^2-5x^2-4)(y^2-5x^2+4)=25x^4-10x^2y^2+y^4-16$$

essentially by the same method; namely, squaring the fundamental identity

$$L_n^2-5F_n^2=4(-1)^n,$$

which is Hoggatt's identity (I_{12}) from [1].

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Bob Prielipp, H.-J. Seiffert, Indulis Strazdins, and the proposer.

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