PALINDROMIC NUMBERS IN ARITHMETIC PROGRESSIONS

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Integers have many interesting properties. In this paper it will be shown that, for an arbitrary nonconstant arithmetic progression $\{a_n\}_{n=1}^{\infty}$ of positive integers (denoted by $\mathbb{N}$), either $\{a_n\}_{n=1}^{\infty}$ contains infinitely many palindromic numbers or else $10|a_n$ for every $n \in \mathbb{N}$. (This result is a generalization of the theorem concerning the existence of palindromic multiples, cf. [2].) More generally, for any number system base b, a nonconstant arithmetic progression of positive integers contains infinitely many palindromic numbers if and only if there exists a member of the progression not divisible by b.

WHAT IS A PALINDROMIC NUMBER?

A positive integer is said to be a (decadic) palindromic number or, shortly, a palindrome if its leftmost digit is the same as its rightmost digit, its second digit from the left is equal to its second digit from the right, and so on. For example, 33, 142505241, and 6 are palindromic numbers. More precisely, let $\overline{d_k d_{k-1} \ldots d_1 d_0}$ be a usual decadic expansion of n, where $d_i \in \{0, 1, \ldots, 8, 9\}$ for $i \in \{0, 1, \ldots, k\}$ and $d_k \neq 0$. That is, $n = \sum_{i=0}^{k} d_i \cdot 10^i$.

Definition: A positive integer n is called a palindrome if its decadic expansion $n = \sum_{i=0}^{k} d_i \cdot 10^i$ satisfies $d_i = d_{k-i}$ for all $i \in \{0, 1, \ldots, k\}$.

In Harminc's paper [2], interesting properties of palindromes were observed. For instance, a palindromic number is divisible by 81 if and only if the sum of its digits is divisible by 81. Some open questions were also stated there. For example, it is not known whether there exist infinitely many palindromic primes. Korec has proved in [3] and [4] that there are infinitely many non-palindromic numbers having palindromic squares.

In what follows we will consider arithmetic progressions. Each such progression $\{a_n\}_{n=1}^{\infty}$ is given by its first member a_1 and by its difference d; thus, $a_n = a_1 + (n-1) \cdot d$. Let us recall a well-known result on prime numbers in arithmetic progressions proved by Dirichlet (cf. [1]). As usual, denote by (u, v) the greatest common divisor of integers u and v. If $(u, v) = 1$, then u and v are called pairwise prime integers. Integers a and b are said to be congruent modulo a positive integer m, if $m|(a-b)$; for this, we will write $a \equiv b \pmod{m}$. Then, the theorem of Dirichlet is

Theorem A: Every arithmetic progression in which the first member and the common difference are pairwise prime integers has infinitely many primes.

In other words, if $(a_1, d) = 1$, then the congruence $x \equiv a_1 \pmod{d}$ has infinitely many prime solutions. We will present an analogous result giving easy necessary and sufficient conditions for an arithmetic progression to contain infinitely many palindromic solutions. Clearly, if every member of an arithmetic progression ends in zero, then the progression cannot contain any palindromic number. But as we will see, this is the unique exception.

MULTIPLES OF THE TYPE 999...99

Lemma: Let $e \in \mathbb{N}$ be such that $(e, 10) = 1$. Then, for every $m_0 \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m > m_0$ and $10^m \equiv 1 \pmod{e}$.

Proof: Let us investigate powers of ten. Each number 10^k ($k \in \mathbb{N}$) is congruent (mod e) to one of the numbers $0, 1, 2, \ldots, e-1$. From this fact, it follows that, among the powers 10, 10^2, ..., 10^i, ..., there exist infinitely many numbers pairwise congruent (mod e). Thus, there are $k_1, k_2 \in \mathbb{N}$ such that

$$10^{k_1} \equiv 10^{k_2} \pmod{e} \quad \text{and} \quad k_2 - k_1 > m_0.$$

Then $10^{k_1} \cdot (1 - 10^{k_2 - k_1}) \equiv 0 \pmod{e}$ and, since $(e, 10) = 1$, we obtain $10^{k_2 - k_1} \equiv 1 \pmod{e}$. Hence, $m = k_2 - k_1$ has the desired properties. □

Since $10^m \equiv 1 \pmod{e}$ means that $e \mid \underbrace{999\ldots99}_{m \text{ of 9's}}$, the Lemma yields the following corollary.

Corollary: If $e \in \mathbb{N}$ and $(e, 10) = 1$, then there exist infinitely many numbers of the type 999...99 divisible by e.

MAIN RESULT

Before stating Theorem B, let us introduce a notation used in the proof. An integer with the same digits as $n \in \mathbb{N}$, but written in the opposite order, will be denoted by n^*, i.e., if $n = \overline{d_k d_{k-1} \ldots d_1 d_0}$, then $n^* = \overline{d_0 d_1 \ldots d_{k-1} d_k}$. Thus, n is a palindrome if and only if $n = n^*$.

Theorem B: Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic progression of positive integers with difference $d \in \mathbb{N}$. Then $\{a_n\}_{n=1}^{\infty}$ contains infinitely many palindromes if and only if $10 \nmid a_1$ or $10 \nmid d$.

Proof: Clearly, if there exists $i \in \mathbb{N}$ such that a_i is a palindrome, then a_i and d cannot both be multiples of ten.

Conversely, let $10 \nmid a_1$ or $10 \nmid d$ and let $d = 2^{\beta} \cdot 5^{\gamma} \cdot e$, where $(e, 10) = 1$. Let us denote by c the least member of the sequence $\{a_n\}_{n=1}^{\infty}$ that is not divisible by 10 and let

$$c = \overline{c_t c_{t-1} \ldots c_1 c_0} = \sum_{i=0}^{t} c_i \cdot 10^i$$

where $c_t \neq 0$. (Since $10 \nmid a_1$ or $10 \nmid d$, we have $c = a_1$ or $c = a_2$.)

Consider two cases, $e = 1$ and $e \neq 1$. The idea is (in the first case) to insert a sufficiently large number of 0's between c^* and c (in the second case) to include among the 0's an appropriate number of strategically placed 1's.

First, let $e = 1$. Then, for every integer $l > \max\{t, \beta, \gamma\}$, it is easy to see that the palindrome

$$c^* \cdot 10^l + c = \overline{c_0 c_1 \ldots c_{t-1} c_t \underbrace{0 \ldots 0}_{l-t-1 \text{ of 0's}} c_t c_{t-1} \ldots c_1 c_0}$$

is a member of the sequence $\{a_n\}_{n=1}^{\infty}$.

Now, let $e \neq 1$. By the Lemma above, there exists $m > \max\{t, \beta, \gamma\}$ such that $10^m \equiv 1$ (mod e). Then, for every integer $j \in \mathbb{N}$, we have $10^{jm} \equiv 1$ (mod e). Moreover, there exists $r \in \{0, 1, \ldots, e-1\}$ such that

$$c^* \cdot 10^{m-t} + r \equiv 0 \pmod{e}.$$

Put

$$x = c^* \cdot 10^{rm+m-t} + 10^{rm} + 10^{(r-1)m} + \cdots + 10^m + c$$

$$= \overline{c_0 c_1 \ldots c_{t-1} c_t \underbrace{0\ldots0}_{m-t-1} 1 \underbrace{0\ldots0}_{m-1} 1 \underbrace{0\ldots0}_{m-1} 10\ldots01 \underbrace{0\ldots0}_{m-1} 1 \underbrace{0\ldots0}_{m-t-1} c_t c_{t-1} \ldots c_1 c_0}.$$

Clearly, x is a palindrome, and we will show that x is a member of the sequence $\{a_n\}_{n=1}^{\infty}$. Therefore, it is sufficient to check that $d \mid (x-c)$. Since $2^{\beta} \mid (x-c)$ and $5^{\gamma} \mid (x-c)$, we will verify only that $e \mid (x-c)$. But

$$x - c = c^* \cdot 10^{rm+m-t} + 10^{rm} + 10^{(r-1)m} + \cdots + 10^m$$

$$\equiv c^* \cdot 10^{rm} \cdot 10^{m-t} + \underbrace{1 + \cdots + 1}_{r} \pmod{e}.$$

Hence

$$x - c \equiv c^* \cdot 10^{m-t} + r \pmod{e},$$

so that $x-c$ is congruent to zero (mod e), and the proof if complete. □

One could define a b-adic palindrome as a positive integer n with b-adic expansion $\overline{d_k d_{k-1} \ldots d_1 d_0}$ (i.e., $n = \sum_{i=0}^{k} d_i \cdot b^i$, where $d_i \in \{0, 1, \ldots, b-1\}$ and $d_k \neq 0$) satisfying $d_i = d_{k-i}$ for all $i \in \{0, 1, \ldots, k\}$. It is not difficult to see that all results proved here for decadic palindromes hold for b-adic ones, too. For any number system base b, the following theorem is true.

Theorem C: Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic progression of positive integers with difference $d \in \mathbb{N}$. Then $\{a_n\}_{n=1}^{\infty}$ contains infinitely many b-adic palindromes if and only if $b \nmid a_1$ or $b \nmid d$.

To prove Theorem C, the reader can mimic the proof of Theorem B.

Hint: Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ be the standard form of b and let $d = p_1^{\beta_1} p_2^{\beta_2} \ldots p_s^{\beta_s} \cdot e$, where $(e, b) = 1$. Let c be as before with

$$c = \overline{c_t c_{t-1} \ldots c_1 c_0} = \sum_{i=0}^{t} c_i \cdot b^i.$$

If $e = 1$, take $x = c^* \cdot b^l + c$, where

$$l > \max\left\{t, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_s}{\alpha_s}\right\}.$$

If $e \neq 1$, take $x = c^* \cdot b^{rm+m-t} + b^{rm} + b^{(r-1)m} + \cdots + b^m + c$, where m is sufficiently large, see the Lemma above for

$$m_0 \geq \max\left\{t, \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_s}{\alpha_s}\right\}. \quad \square$$

Open problem: Characterize geometric progressions without palindromic members.

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