# CONGRUENCES MOD $p^{\boldsymbol{n}}$ FOR THE BERNOULLI NUMBERS 

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## 1. INTRODUCTION

Let $p$ be a prime. In 1889 Voronoi proved the congruence

$$
\begin{equation*}
\left(a-a^{p-2 k}\right) \frac{B_{2 k}}{2 k} \equiv \sum_{s=1}^{p-1}\left[\frac{s a}{p}\right] s^{2 k-1}(\bmod p), \tag{1}
\end{equation*}
$$

where $k, a$ are positive integers such that $p$ does not divide $a$ and $p-1$ does not divide $2 k ; B_{2 k}$ is the $2 k^{\text {th }}$ Bernoulli number. More general versions of this congruence can be found in [6] or [3]. Following Wagstaff, denote congruence (1) also by the symbol $\{a\}$. Adding together congruences $\{2\}$, $\{3\}$, and $-\{4\}$, we obtain the congruence

$$
\{2\}+\{3\}-\{4\}
$$

which, after some obvious cancellations in the right member, takes the form

$$
\begin{equation*}
\left(2^{p-2 k}+3^{p-2 k}-4^{p-2 k}-1\right) \frac{B_{2 k}}{4 k} \equiv \sum_{p / 4<s<p / 3} s^{2 k-1}(\bmod p), \tag{2}
\end{equation*}
$$

provided that $p>4$. Several such identities are also obtainable in a way analogous to that shown above by using suitable variations of parameter $a$. Several authors used formulas of this type to test regularity via computer. The best result in this direction is the following one, due to Tanner and Wagstaff [5], which is valid for all primes $p>10$,

$$
\begin{align*}
& \left(2^{p-2 k}+9^{p-2 k}-10^{p-2 k}-1\right) \frac{B_{2 k}}{4 k} \equiv\left(1+2^{2 k-1}+3^{2 k-1}+4^{2 k-1}\right) \sum_{\frac{p}{10}<s<\frac{13 p}{120}} s^{2 k-1} \\
& \quad+\left(1+2^{2 k-1}+3^{2 k-1}+4^{2 k-1}+12^{2 k-1}\right) \sum_{\frac{13 p}{120}<s<\frac{p}{9}} s^{2 k-1} \\
& \quad-3^{2 k-1} \sum_{\frac{2 p}{9}<s<\frac{7 p}{30}} s^{2 k-1}-\left(2^{2 k-1}+6^{2 k-1}\right) \sum_{\frac{5}{18}<s<\frac{17 p}{60}} s^{2 k-1}  \tag{3}\\
& \quad-2^{2 k-1} \sum_{\frac{17 p}{60} \lll \frac{3 p}{10}} s^{2 k-1}-\left(2^{2 k-1}+4^{2 k-1}+12^{2 k-1}\right) \sum_{\frac{7 p}{18}<s<\frac{4 p}{120}} s^{2 k-1} \\
& \quad-\left(2^{2 k-1}+4^{2 k-1}\right) \sum_{\frac{47 p}{120}<s<\frac{2 p}{5}} s^{2 k-1}(\bmod p) .
\end{align*}
$$

In formula (3), the sums in the right member contain a total of about $p / 18$ terms [formula (2) contains about $p / 12$ terms while formula (1) contains $(p-1) / 2$ terms for $a=2$ ]. All the applications of these formulas concerning Fermat's Last Theorem are now mainly of historical interest
after Wiles's proof [8] of FLT. There are congruences of various types for the Bernoulli numbers. Recent results on congruences for Bernoulli numbers of higher order can be found in [2].

We shall prove the following analog of formula (1).
Theorem 1: Let $\chi$ be a primitive Dirichlet character with modulus $m \geq 2$. If $a \geq 2$ is an integer such that $m$ does not divide $a$, then

$$
\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s)= \begin{cases}0 & \text { if } \chi \text { is even }  \tag{4}\\ -\frac{\bar{\chi}(a)-a}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s) & \text { if } \chi \text { is odd }\end{cases}
$$

where the bar means complex conjugation.
The proof of Theorem 1 will be given in Section 2. Formula (4) can be written, equivalently, in the form

$$
\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s)= \begin{cases}0 & \text { if } \chi \text { is even }  \tag{5}\\ \frac{a-\bar{\chi}(a)}{m} \sum_{s=1}^{m-1} s \chi(s) & \text { if } \chi \text { is odd }\end{cases}
$$

because of the formula

$$
\begin{equation*}
\sum_{s=1}^{m-1} s \chi(s)=\frac{m}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s) \tag{6}
\end{equation*}
$$

which is valid for an odd primitive character $\chi$.
We use formula (5) to obtain $p^{n}$-divisibility criteria for Bernoulli numbers of the form

$$
B_{(2 k-1) p^{n}+1}, \quad k=1,2, \ldots, \frac{p-3}{2}
$$

Criteria of this type are still of interest because of their connection with the invariants of the irregular class group of a properly irregular cyclotomic field [7] (cf. also [4], p. 189). Assume now that $m=p$, an odd prime. Let $\psi$ be the character defined as the $p$-adic limit

$$
\psi(s)=\lim _{n \rightarrow \infty} s^{p^{n}}
$$

for every $s$ prime to $p$. All the values of $\psi$ belong to $\mathbb{Z}_{p}$, the ring of $p$-adic integers. Moreover,

$$
\psi(s) \equiv s^{p^{n-1}}\left(\bmod p^{n}\right), \quad n \geq 1
$$

For an odd character, we have $\chi=\psi^{2 k-1}$, for some $k \geq 1$, and

$$
\begin{gathered}
\chi(s) \equiv s^{(2 k-1) p^{n-1}}\left(\bmod p^{n}\right) \\
\bar{\chi}(s) \equiv s^{-p^{n-1}(2 k-1)} \equiv s^{p^{n-1}(p-1)-p^{n-1}(2 k-1)} \equiv s^{p^{n-1}(p-2 k)}\left(\bmod p^{n}\right)
\end{gathered}
$$

Theorem 2: Let $p$ be a prime $>3$. If $a$ is an integer such that $p$ does not divide $a$, then

$$
\begin{equation*}
\left[a-a^{p^{n-1}(p-2 k)}\right] B_{(2 k-1) p^{n}+1} \equiv \sum_{s=1}^{p-1}\left[\frac{s a}{p}\right] s^{(2 k-1) p^{n-1}}\left(\bmod p^{n}\right) \tag{7}
\end{equation*}
$$

for every $k \geq 1$ such that $p-1$ does not divide $2 k$.

Proof: We consider the $n^{\text {th }}$ Bernoulli polynomial

$$
B_{n}(x)=\sum_{j=0}^{n}\binom{n}{j} B_{j} x^{n-j}, n \geq 1 .
$$

Then, for the odd character $\chi=\psi^{2 k-1}$, we have

$$
\begin{aligned}
\sum_{s=1}^{p-1} s \chi(s) & \equiv \sum_{s=1}^{p} s^{(2 k-1) p^{n}+1} \equiv \frac{B_{(2 k-1) p^{n}+2}(p)-B_{(2 k-1) p^{n}+2}}{(2 k-1) p^{n}+2} \\
& \equiv p B_{(2 k-1) p^{n}+1}+\frac{\left[(2 k-1) p^{n}+1\right](2 k-1) p^{n}}{3!} p^{3} B_{(2 k-1) p^{n}-1}+\cdots \\
& \equiv p B_{(2 k-1) p^{n}+1}\left(\bmod p^{n+1}\right) .
\end{aligned}
$$

Since $p-1$ does not divide $2 k$, we obtain the congruence

$$
\frac{1}{p} \sum_{s=1}^{p-1} s \chi(s) \equiv B_{(2 k-1) p^{n}+1}\left(\bmod p^{n}\right),
$$

which, together with Theorem 1 and relation (5), yields the sought result.
For $n=1$, congruence (7) reduces to congruence (1) since

$$
B_{(2 k-1) p+1}=[(2 k-1) p+1] \frac{B_{(2 k-1) p+1}}{(2 k-1) p+1} \equiv \frac{B_{2 k}}{2 k}(\bmod p)
$$

because of Kummer's congruence.
We can prove, using exactly analogous techniques and starting from (7), a $p^{n}$-analog of congruence (3). Because of the obvious analogy between the proofs, the sought result follows simply by replacing expressions of the form

$$
a^{p-2 k}, a^{2 k-1}, s^{2 k-1}, \frac{B_{2 k}}{2 k}
$$

in congruence (3) with the respective expressions

$$
a^{p^{n-1}(p-2 k)}, a^{(2 k-1) p^{n-1}}, s^{(2 k-1) p^{n-1}}, B_{(2 k-1) p^{n}+1} .
$$

The following theorem then follows.
Theorem 3: Let $p$ be an odd prime $>10, k \geq 1, p-1$ does not divide $2 k$ and $n \geq 1$. Then

$$
\begin{aligned}
& \frac{2^{(p-2 k) p^{n-1}}+9^{(p-2 k) p^{n-1}}-10^{(p-2 k) p^{n-1}}-1}{2} B_{(2 k-1) p^{n+1}} \\
& \equiv\left[1+2^{\left.(2 k-1) p^{n-1}+3^{(2 k-1) p^{n-1}}+4^{(2 k-1) p^{n-1}}\right] \sum_{\frac{p}{10}<s<\frac{13 p}{120}} s^{(2 k-1) p^{n-1}}}\right. \\
& \quad+\left[1+2^{(2 k-1) p^{n-1}}+3^{(2 k-1) p^{n-1}}+4^{(2 k-1) p^{n-1}}+12^{(2 k-1) p^{n-1}}\right] \sum_{\frac{13 p}{120<s<\frac{p}{9}}} s^{(2 k-1) p^{n-1}} \\
& \quad-3^{(2 k-1) p^{n-1}} \sum_{\frac{2 p}{9<s<\frac{7 p}{30}}} s^{(2 k-1) p^{n-1}}-\left[2^{(2 k-1) p^{n-1}}+6^{(2 k-1) p^{n-1}}\right] \sum_{\frac{5 p}{18<s<\frac{17 p}{60}}} s^{(2 k-1) p^{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
& -2^{(2 k-1) p^{n-1}} \sum_{\frac{17}{60}<s<\frac{3 p}{10}} s^{(2 k-1) p^{n-1}}-\left[2^{(2 k-1) p^{n-1}}+4^{(2 k-1) p^{n-1}}+12^{(2 k-1) p^{n-1}}\right] \sum_{\frac{7 p}{18}<s<\frac{47 p}{120}} s^{(2 k-1) p^{n-1}} \\
& -\left[2^{(2 k-1) p^{n-1}}+4^{(2 k-1) p^{n-1}}\right] \sum_{\frac{47 p}{120}<s<\frac{2 p}{5}} s^{(2 k-1) p^{n-1}}\left(\bmod p^{n}\right) .
\end{aligned}
$$

The congruence contains in the right member $p / 18$ terms only.

## 2. PROOF OF THEOREM 1

At first, we note that, obviously,

$$
\begin{equation*}
-\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s)=\sum_{j=1}^{a} \sum_{s=0}^{[j m / a]} \chi(s) . \tag{8}
\end{equation*}
$$

For integer $j, 0<j \leq a$, define

$$
\Phi(x)= \begin{cases}\frac{1}{2} & \text { if } x=0 \text { or } 2 \pi j / a \\ 1 & \text { if } 0<x<2 \pi j / a \\ 0 & \text { if } 2 \pi j / a<x<2 \pi\end{cases}
$$

and continue $\Phi(x)$ periodically with period $2 \pi$ over the real numbers. The function $\Phi(x)$ has the Fourier expansion

$$
\Phi(x)=\sum_{n=-\infty}^{\infty} c_{n} e^{i n x} \quad(i=\sqrt{-1})
$$

where

$$
c_{n}=\frac{1}{2 \pi} \int_{0}^{2 \pi} \Phi(x) e^{-i n x} d x=\frac{i}{2 \pi n}\left(e^{-\frac{2 \pi i j n}{a}}-1\right) .
$$

First, we assume that $a<m$. Then

$$
\begin{aligned}
\sum_{s=0}^{[j m / a]} \chi(s) & =\sum_{s=1}^{m-1} \chi(s) \Phi\left(\frac{2 \pi s}{m}\right) \\
& =\frac{i}{2 \pi} \sum_{s=1}^{m-1} \chi(s) \sum_{n=-\infty}^{\infty} \frac{\left(e^{-\frac{2 \pi i n}{a}}-1\right) e^{\frac{2 \pi i s}{m}}}{n} \\
& =\frac{i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{2 \pi i n}{a}}-1}{n} \sum_{s=1}^{m-1} \chi(s) e^{\frac{2 \pi i s}{m}} \\
& =\frac{\tau(\chi) i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\left(e^{-\frac{2 \pi j n}{a}}-1\right) \bar{\chi}(n)}{n},
\end{aligned}
$$

where

$$
\tau(\chi)=\sum_{s=1}^{m-1} \chi(s) e^{\frac{2 \pi i s}{m}}
$$

As a consequence,

$$
\begin{aligned}
\sum_{j=1}^{a} \sum_{s=0}^{[j m / a]} \chi(s) & =\frac{\tau(\chi) i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n)}{n}\left(\sum_{j=1}^{a} e^{-\frac{2 \pi j i n}{a}}-a\right) \\
& =\frac{\tau(\chi) i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{j=1}^{a} e^{-\frac{2 \pi i j n}{a}}-\frac{\tau(\chi) i a}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n)}{n} .
\end{aligned}
$$

Since

$$
\sum_{j=1}^{a} e^{-\frac{2 \pi j i n}{a}}= \begin{cases}l & \text { if } n \equiv 0(\bmod a), \\ 0 & \text { if } n \not \equiv 0(\bmod a),\end{cases}
$$

it follows that

$$
\begin{aligned}
\sum_{j=1}^{a} \sum_{s=0}^{[j m / a]} \chi(s) & =\frac{\tau(\chi) i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n a)}{n a} a-\frac{\tau(\chi) i}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n)}{n} \\
& =\frac{\tau(\chi) i}{2 \pi}(\bar{\chi}(a)-a) \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(n)}{n} .
\end{aligned}
$$

For even $\chi$, the last infinite sum is equal to zero while, for odd $\chi$, it is equal to $2 L(1, \bar{\chi})$. In view of the formula (cf. [1], p. 336)

$$
L(1, \bar{\chi})=\frac{\pi i}{(2-\bar{\chi}(2)) \tau(x)} \sum_{s=1}^{[m / 2]} \chi(s)
$$

and relation (8), it follows that

$$
\begin{equation*}
\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s)=-\frac{\bar{\chi}(a)-a}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s) \tag{9}
\end{equation*}
$$

for $a<m$. It remains to prove the theorem for $a>m$. Then $a=a_{1}+m t$, where $a_{1}$ and $t$ are integers and $0<a_{1}<m$. Also $m$ does not divide $a_{1}$. We have

$$
\begin{aligned}
\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s) & =\sum_{s=1}^{m-1}\left[\frac{s a_{1}}{m}+s t\right] \chi(s) \\
& =\sum_{s=1}^{m-1}\left[\frac{s a_{1}}{m}\right] \chi(s)+t \sum_{s=1}^{m-1} s \chi(s) .
\end{aligned}
$$

The last expression is zero for even $\chi$. For odd $\chi$ we have, in view of (6) and (9),

$$
\begin{aligned}
\sum_{s=1}^{m-1}\left[\frac{s a}{m}\right] \chi(s) & =-\frac{\bar{\chi}\left(a_{1}\right)-a_{1}}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s)+\frac{t m}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s) \\
& =-\frac{\bar{\chi}(a)-\left(a_{1}+t m\right)}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s) \\
& =-\frac{\bar{\chi}(a)-a}{\bar{\chi}(2)-2} \sum_{s=1}^{[m / 2]} \chi(s),
\end{aligned}
$$

which proves the theorem for $a>m$.

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