ARITHMETIC FUNCTIONS OF FIBONACCI NUMBERS

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(Submitted November 1997-Final Revision February 1998)

For any integers $n \geq 1$ and $k \geq 0$, let $\phi(n)$ and $\sigma_k(n)$ be the Euler totient function of n and the sum of the k^{th} powers of the divisors of n, respectively. In this note, we present the following inequalities.

Theorem:

(1) $\phi(F_n) \geq F_{\phi(n)}$ for all $n \geq 1$. Equality is obtained only if $n = 1, 2, 3$.

(2) $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ for all $n \geq 1$ and $k \geq 1$. Equality is obtained only if $n = 1$ or $(k, n) = (1, 3)$.

(3) $\sigma_0(F_n) \geq F_{\sigma_0(n)}$ for all $n \geq 1$. Equality is obtained only if $n = 1, 2, 4$.

Proof:

(1) See [2] for a more general result. □

(2) Let $k \geq 1$. Notice that $\sigma_k(F_1) = F_{\sigma_k(1)} = 1$ for all $k \geq 1$. Moreover, as $\sigma_k(2) = 1 + 2^k \geq 3$ for $k \geq 1$, it follows that $F_{\sigma_k(2)} = F_{1+2^k} \geq F_3 = 2 > 1 = \sigma_k(1) = \sigma_k(F_2)$. Now let $n = 3$. Notice that $F_{\sigma_1(3)} = F_4 = 3 = \sigma_1(2) = \sigma_1(F_3)$. However, if $k \geq 2$, then $\sigma_k(3) = 1 + 3^k \geq 10$. Since $F_n > n$ for $n \geq 6$, it follows that $F_{\sigma_k(3)} = F_{1+3^k} > 1 + 3^k > 1 + 2^k = \sigma_k(2) = \sigma_k(F_3)$ for $k \geq 2$. From this point on, we assume that $n \geq 4$.

Moreover, assume that

$$\sigma_k(F_n) \geq F_{\sigma_k(n)} \tag{1}$$

for some $n \geq 4$ and some $k \geq 1$. First, we show that if inequality (1) holds, then n is prime. Indeed, assume that n is not prime.

Since $n^k \geq nk$ for all $n \geq 4$ and $k \geq 1$, and since $F_{u+v} \geq F_u \cdot F_v$ for all integers u and v, it follows that

$$F_{n^k} \geq F_{nk} \geq F_n^k \quad \text{for } n \geq 4 \text{ and } k \geq 1. \tag{2}$$

Clearly

$$\frac{m}{\phi(m)} > \frac{\sigma_k(m)}{m^k} \quad \text{for } m \geq 2 \text{ and } k \geq 1. \tag{3}$$

If $n \leq 41$, then $F_n \leq F_{41} < 2 \cdot 10^9$. By Lemma 4.2 in [3], it follows that

$$6 > \frac{F_n}{\phi(F_n)}, \tag{4}$$

and by inequalities (1)-(4), it follows that

$$F_6 = 8 > 6 > \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_{n^k}} \geq F_{\sigma_k(n)-n^k}. \tag{5}$$

Hence, $6 > \sigma_k(n) - n^k$. Since n is not prime, it follows that

$$\sigma_k(n) - n^k \geq \sqrt{n}^k. \tag{6}$$

Therefore, $6 > \sqrt{n}^k$. Since $n \geq 4$, it follows that $6 > \sqrt{4}^k = 2^k$ or $k < 3$. The only pairs (k, n) satisfying the inequality $6 > \sqrt{n}^k$ for which $4 \leq n \leq 40$ is not prime are $(k, n) = (2, 4)$ and $(1, n)$, where $4 \leq n \leq 35$ is not prime. One can check using Mathematika, for example, that $F_{\sigma_k(n)} > \sigma_k(F_n)$ for all the above pairs (k, n).

Suppose now that inequality (1) holds for some $k \geq 1$ and some $n \geq 42$ that is not a prime. Since $F_n \geq F_{42} > 2 \cdot 10^9$, it follows by Lemma 4.1 in [3] that

$$\log(F_n) > \frac{F_n}{\phi(F_n)}. \tag{7}$$

By inequalities (1), (2), (3), and (7), it follows that

$$\log(F_n) > \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_{n^k}} \geq F_{\sigma_k(n)-n^k}. \tag{8}$$

Since

$$\left(\frac{1+\sqrt{5}}{2}\right)^n > F_n > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right) \quad \text{for all } n \geq 1, \tag{9}$$

it follows from inequalities (6) and (9) that

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \log F_n > F_{\sigma_k(n)-n^k} > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{n}^k} - 1\right). \tag{10}$$

If $k \geq 2$, then $\sqrt{n}^k \geq n$, and inequality (10) implies that

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right). \tag{11}$$

Inequality (11) implies that $n < 3$, which contradicts the fact that $n \geq 42$. Hence $k = 1$. Inequality (10) becomes

$$n \log\left(\frac{1+\sqrt{5}}{2}\right) > \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{n}} - 1\right),$$

which implies that $n < 92$. One can check using Mathematika, for example, that $F_{\sigma_1(n)} > \sigma_1(F_n)$ for all $42 \leq n \leq 91$.

From the above arguments, it follows that if inequality (1) holds for some $n \geq 4$ and some $k \geq 1$, then n is prime. In particular, $n \geq 5$,

Write $F_n = q_1^{\gamma_1} \cdot \dots \cdot q_t^{\gamma_t}$, where $q_1 < \dots < q_t$ are prime numbers, and $\gamma_i \geq 1$ for $i = 1, \dots, t$. We show that q_1, q_2, and t satisfy the following conditions:

(a) $q_1 \geq n$;

(b) If $t > 1$, then $q_2 \geq 2n - 1$;

(c) $t - 1 > 2(n-1)\log\left(\frac{3}{2} \cdot e^{-1/(n-1)}\right)$.

Indeed, let q be one of the primes dividing F_n, From Lemma II and Theorem XII in [1], it follows that $q|F_{q^2} \cdot F_{q^2-1}$.

Suppose first that $q|F_{q^2}$. We conclude that $q|(F_n, F_{q^2}) = F_{(n,q^2)}$. Since $F_1 = 1$, we conclude that $(n, q^2) \neq 1$. Since both q and n are prime, it follows that $q = n$.

Suppose now that $q|F_{q^2-1}$. We conclude that $q|(F_n, F_{q^2-1}) = F_{(n,q^2-1)} \cdot q \equiv \pm 1 \pmod{n}$. Then, clearly, $q \neq n \pm 1$ because q and n are both prime and $n \geq 5$. Hence, $q \geq 2n-1$ in this case.

Now (a) and (b) follow immediately from the above arguments.

For (c), notice that by inequalities (1), (2), and (3),

$$\prod_{i=1}^{t}\left(1+\frac{1}{q_i-1}\right) = \frac{F_n}{\phi(F_n)} > \frac{\sigma_k(F_n)}{F_n^k} \geq \frac{F_{\sigma_k(n)}}{F_n^k} = \frac{F_{1+n^k}}{F_n^k} \geq \frac{F_{1+n^k}}{F_{n^k}} \geq \frac{3}{2}, \tag{12}$$

because $F_{m+1}/F_m \geq 3/2$ for all $m \geq 3$. Taking logarithms in inequality (12), and using the fact that $\log(1+x) < x$ for all $x > 0$, we conclude that

$$\sum_{i=1}^{t}\frac{1}{q_i-1} > \log\left(\frac{3}{2}\right).$$

From (a) and (b), it follows that

$$\frac{1}{n-1} + \frac{t-1}{2(n-1)} > \log\left(\frac{3}{2}\right). \tag{13}$$

Inequality (13) is obviously equivalent to the inequality asserted at (c) above.

From inequality (10) and inequalities (a)-(c) above, it follows that

$$n\log\left(\frac{1+\sqrt{5}}{2}\right) > \log F_n \geq \sum_{i=1}^{t}\log q_i \geq \log n + (t-1)\log(2n-1)$$
$$> (t-1)\log(2n-1) > 2(n-1)\log(2n-1)\log\left(\frac{3}{2}\cdot e^{-1/(n-1)}\right).$$

Hence,

$$\frac{n}{2(n-1)\log(2n-1)} \cdot \log\left(\frac{1+\sqrt{5}}{2}\right) - \log\left(\frac{3}{2}\right) + \frac{1}{n-1} > 0. \tag{14}$$

Inequality (14) implies that $n < 5$, which contradicts the fact that $n \geq 5$. □

(3) Let $k = 0$. For any positive integer m, let $\tau(m)$ and $\nu(m)$ be the number of divisors of m and the number of prime divisors of m, respectively. Notice that $\tau(m) = \sigma_0(m)$. Therefore, the inequality asserted at (3) is equivalent to $\tau(F_n) \geq F_{\tau(n)}$ for $n \geq 1$.

Let n be a positive integer. Recall that a *primitive divisor* of F_n is a prime number q, such that $q|F_n$, but $q \nmid F_m$ for any $1 \leq m < n$. From Theorem XXIII in [1], we know that F_n has a primitive divisor for all $n \geq 1$ except $n = 1, 2, 6, 12$. We distinguish the following cases.

Case 1. $6 \nmid n$. Since $F_d|F_n$ for all $d|n$, and F_d has a primitive divisor for all d except $d = 1$, 2, it follows that $\nu(F_n) \geq \tau(n) - 2$. Hence,

$$\tau(F_n) \geq 2^{\nu(F_n)} \geq 2^{\tau(n)-2}. \tag{15}$$

Since $2^{k-2} > F_k$ for all $k \geq 4$, it follows that the inequality asserted by (3) holds for all n such that $\tau(n) \geq 4$.

If $\tau(n) = 1$, then $n = 1$ and $\tau(F_1) = F_{\tau(1)} = 1$.

If $\tau(n) = 2$, then $n = p$ is a prime and $\tau(F_p) \geq 1 = F_2 = F_{\tau(p)}$. Obviously, equality holds only if $p = 2$.

If $\tau(n) = 3$, then $n = p^2$, where p is a prime. Moreover, $\tau(F_{p^2}) \geq 2 = F_3 = F_{\tau(p^2)}$, and equality certainly holds for $p = 2$. If $p > 2$, then both F_p and F_{p^2} have a primitive divisor; therefore,

$$\tau(F_{p^2}) \geq 4 > 2 = F_3 = F_{\tau(p^2)}.$$

Case 2. $6 \mid n$, but $12 \nmid n$. In this case, $\nu(F_n) \geq \tau(n) - 3$. Moreover, since $F_6 = 8 \mid F_n$, it follows that the exponent at which 2 appears in the prime factor decomposition of F_n is at least 3. Hence,

$$\tau(F_n) \geq 2^{\nu(n)-1} \cdot (3+1) \geq 2^{\tau(n)-4} \cdot 4 = 2^{\tau(n)-2} > F_{\tau(n)},$$

because $\tau(n) \geq 4 = \tau(6)$.

Case 3. $12 \mid n$. In this case, $\nu(F_n) \geq \tau(n) - 4$. Moreover, since $2^4 \cdot 3^2 = F_{12} \mid F_n$, it follows that the exponents at which 2 and 3 appear in the prime factor decomposition of F_n are at least 4 and 2, respectively. Thus,

$$\tau(F_n) \geq 2^{\nu(n)-2} \cdot (4+1) \cdot (2+1) \geq 2^{\tau(n)-6} \cdot 15. \tag{16}$$

Moreover, since $12 \mid n$, it follows that $\tau(n) \geq 6 = \tau(12)$. By inequality (15), it follows that it suffices to show that

$$15 \cdot 2^{k-6} > F_k \quad \text{for } k \geq 6. \tag{17}$$

This can be proved easily by induction. □

This completes the proof of the Theorem. □

REFERENCES

1. R. D. Carmichael. "On the Numerical Factors of Arithmetic Forms." *Ann. of Math.* **15** (1913-1914):30-70.
2. F. Luca. "Euler Indicators of Lucas Sequences." To appear in *Bull. Math. Soc. Sci. Math. Roumanie.*
3. M. Ward. "The Intrinsic Divisors of Lehmer Numbers." *Ann. of Math.* **62** (1955):230-236.

AMS Classification Numbers: 11A25, 11B39

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