FIBONACCI NUMBERS AND HARMONIC QUADRUPLES

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Here, we combine number theory (Fibonacci numbers) and projective geometry (harmonic fourth).

Let the real numbers

$$A < B < C < D \tag{1}$$

form a harmonic quadruple (see [1], pp. 159-60), i.e.,

$$\frac{B-C}{B-A} : \frac{D-C}{D-A} \text{ (cross ratio) } = -1$$

or

$$D(2B-A-C) = BC - 2CA + AB. \tag{2}$$

The number D is also called a harmonic fourth. The affine map $x \mapsto \alpha x + \beta$ with real numbers $\alpha > 0$ and β does not change equations (1) and (2). Especially, with $\alpha = 2/(C-A)$ and $\beta = -(C+A)/(C-A)$, we get $A_1 = -1$, $C_1 = 1$ and, therefore, $B_1 D_1 = 1$, $0 < B_1 < 1 < D_1$. Then, $B_1 = (2B-A-C)/(C-A) > 0$ implies, from (1), that

$$2B > A + C. \tag{3}$$

It is easy to find harmonic quadruples of squares and also of primes like

$$1^2 < 3^2 < 4^2 < 11^2, \quad 1^2 < 11^2 < 15^2 < 41^2,$$
$$3^2 < 11^2 < 13^2 < 17^2, \quad 4^2 < 9^2 < 11^2 < 17^2,$$

and

$$3 < 11 < 17 < 59, \quad 3 < 23 < 41 < 383,$$
$$5 < 13 < 19 < 61, \quad 7 < 19 < 29 < 139;$$

also, the number 0 together with any three consecutive terms $(n+2)^{-1}$, $(n+1)^{-1}$, n^{-1} of the harmonic series form a harmonic quadruple.

Theorem: There are no harmonic quadruples of Fibonacci numbers.

Proof (by contradiction): For integers $2 \le a < b < c < d$, we replace (1) by

$$F_a < F_b < F_c < F_d \tag{4}$$

and (2) by

$$F_d(2F_b - F_a - F_c) = F_b F_c - 2F_c F_a + F_a F_b. \tag{5}$$

By (3), we must have $2F_b > F_a + F_c \ge 1 + F_c$ and, hence, $c = b+1$; however, $2F_b \ge 2 + F_{b+1}$ or $F_{b-2} \ge 2$ holds exactly for $b \ge 5$. Inequality (3) now says $F_{b-2} \ge 1 + F_a$. By $b \ge 5$, this is satisfied exactly for $2 \le a \le b-3$. Consequently, instead of (5), we have to look at

$$F_d(F_{b-2} - F_a) = F_b F_{b+1} - 2F_a F_{b+1} + F_a F_b \tag{6}$$

or

$$F_dF_{b-2} - F_bF_{b+1} = F_a(F_d - 2F_{b+1} + F_b) \tag{7}$$

for $b \geq 5,\ 2 \leq a \leq b-3,\ d \geq b+2$. We observe that

$$F_d - 2F_{b+1} + F_b \geq F_{b+2} - 2F_{b+1} + F_b = F_{b-2} > 0.$$

For $a = 2$ and $a = b-3$, we obtain "$\geq$" and "$\leq$", respectively, in (7) and thus in (6). This means

$$F_d(F_{b-2} - 1) \geq F_bF_{b+1} - 2F_{b+1} + F_b, \tag{8}$$

and

$$F_dF_{b-4} \leq F_bF_{b+1} - 2F_{b-3}F_{b+1} + F_{b-3}F_b. \tag{9}$$

But

$$F_bF_{b+1} - 2F_{b+1} + F_b + F_{b+3}(1 - F_{b-2}) = 2(F_b + (-1)^b) > 0 \quad (b \geq 3)$$

and (8) imply $d \geq b+4$. Furthermore,

$$F_{b+5}F_{b-4} - F_bF_{b+1} + 2F_{b-3}F_{b+1} - F_{b-3}F_b = (18F_b - 11F_{b+1})F_{b+2} > 0 \quad (b \geq 5)$$

and (9) imply $d \leq b+4$. This leaves $d = b+4$.

We found that $b \geq 5$, and $2 \leq a \leq b-3$, and that (4) and (7) can be replaced, respectively, by $F_a < F_b < F_{b+1} < F_{b+4}$ and

$$F_{b+4}F_{b-2} - F_bF_{b+1} = F_a(F_{b+4} - 2F_{b+1} + F_b)$$

or, equivalently,

$$(F_a - 12F_b + 3F_{b+1})(3F_b + F_{b+1}) = -32F_b^2.$$

This implies $(3F_b + F_{b+1}) | 32F_b^2$. Since $1 = (F_{b+1}, F_b) = (3F_b + F_{b+1}, F_b)$, we obtain

$$(3F_b + F_{b+1}) | 32. \tag{10}$$

But, $3F_5 + F_6 = 23$ and $3F_b + F_{b+1} \geq 37\ (b \geq 6)$. Hence (10) is impossible. □

REFERENCE

1. Oswald Veblen & John Wesley Young. *Projective Geometry*. New York-Toronto-London: Blaisdell Publishing Company, 1910, 1938.

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